

PROPERLY EMBEDDED MINIMAL ANNULI IN $\mathbb{H}^2 \times \mathbb{R}$

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ABSTRACT. In this paper we study the moduli space of properly Alexandrov-embedded, minimal annuli in $\mathbb{H}^2 \times \mathbb{R}$ with horizontal ends. We say that the ends are horizontal when they are graphs of $\mathcal{C}^{2,\alpha}$ functions over $\partial_\infty \mathbb{H}^2$. Contrary to expectation, we show that one can not fully prescribe the two boundary curves at infinity, but rather, one can prescribe the bottom curve, but the top curve only up to a translation and a tilt, along with the position of the neck and the vertical flux of the annulus. We also prove general existence theorems for minimal annuli with discrete groups of symmetries.

1. INTRODUCTION

This paper studies the space of properly embedded annuli with horizontal ends in $\mathbb{H}^2 \times \mathbb{R}$. Prototypes of such surfaces are the so-called vertical catenoids C . These are surfaces of revolution with respect to some vertical axis $\{p\} \times \mathbb{R}$. Their asymptotic boundary is the union of two parallel circles in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ and their ends are horizontal in the sense that there are functions u^\pm defined on $\mathbb{H}^2 \setminus K$ for some compact set K such that the ends of C are graphs $t = u^\pm(z)$. They are also symmetric around a horizontal plane $\mathbb{H}^2 \times \{t_0\}$, so in particular, if we translate so that $t_0 = 0$, then $u^-(z) = -u^+(z)$. Here and later, we use the Poincaré disc model $\{|z| < 1\}$ of \mathbb{H}^2 with metric $g_0 = 4|dz|^2/(1 - |z|^2)^2$, so the product metric on $\mathbb{H}^2 \times \mathbb{R}$ is $g = g_0 + dt^2$, and also write $z = re^{i\theta}$, $r < 1$. More generally, we seek minimal annuli $A \subset \mathbb{H}^2 \times \mathbb{R}$ for which the asymptotic boundary $\partial_\infty A$ is a union of two curves γ^\pm which are horizontal in the sense that they can be represented as graphs $t = \gamma^\pm(\theta)$, $\theta \in \mathbb{S}^1 = \partial_\infty \mathbb{H}^2$. For the vertical catenoids described above, these boundary curves are constant graphs, $\gamma^\pm(\theta) \equiv a^\pm$. The general question is to determine which pairs $\Gamma = (\gamma^\pm)$ (initially with $\gamma^-(\theta) \leq \gamma^+(\theta)$, $\theta \in \mathbb{S}^1$) bound a properly embedded minimal annulus with horizontal ends. We often omit the subscript ∞ in the notation for asymptotic boundary below.

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Taking a broader perspective, the asymptotic Plateau problem in $\mathbb{H}^2 \times \mathbb{R}$ asks for a characterization of those curves (or closed subsets) in the asymptotic boundary of $\mathbb{H}^2 \times \mathbb{R}$ which bound complete minimal surfaces. Implicit in this question is a choice of compactification of this space. This question is discussed in some generality in [5]; in the present paper we consider only the product compactification $(\mathbb{H}^2 \times \mathbb{R})^\times = \overline{\mathbb{H}^2} \times \overline{\mathbb{R}}$, which is the product of a closed disk and a closed interval, and only consider boundary curves lying in the vertical part of the boundary $\overline{\mathbb{H}^2} \times \mathbb{R}$. The paper [5] describes a number of different families of examples of ‘admissible’ (connected) boundary curves and notes various obstructions for such curves to be asymptotic boundaries. Related work is contained in the paper [2].

As above, a curve γ is called horizontal if it lies in the vertical boundary $(\partial_\infty \overline{\mathbb{H}^2}) \times \mathbb{R}$ of this product compactification and is a graph $t = \gamma(\theta)$, $\theta \in \mathbb{S}^1$. The simplest problem is to determine whether any connected horizontal curve bounds a minimal surface, and this was settled by Nelli and Rosenberg [9]. They proved that if $\gamma(\theta) \in \mathcal{C}^0(\mathbb{S}^1)$, then there exists a unique function u defined on the disk D , with $u = \gamma$ at $r = 1$, such that the graph of u is minimal in $\mathbb{H}^2 \times \mathbb{R}$. Moreover, this solution is unique, so any complete embedded minimal surface with connected horizontal boundary must be a vertical graph. We refer to [5] and [2] for a list of various general existence results for other classes of connected boundary curves.

The existence result for pairs of horizontal boundary curves, γ^\pm , one lying above the other, is more complicated. As above, we consider only minimal annuli, though certain facts hold even for higher genus surfaces. First, not every pair γ^\pm is fillable by minimal annuli. For example, these curves cannot be too far apart. A simple barrier argument proves that if $\gamma^+(\theta) \geq a + \pi$ and $\gamma^-(\theta) \leq a$ for some a , i.e., there is a band of width π between these curves, then there is no minimal annulus with these asymptotic boundaries and with horizontal ends. It is likely that a stronger result holds, that if $\gamma^+(\theta) - \gamma^-(\theta) \geq \pi$ for all θ , then no such minimal annulus exists, but we do not prove this here.

Define

$$\mathfrak{C} = \{(\gamma^+, \gamma^-) : \gamma^\pm \in \mathcal{C}^{2,\alpha}(\mathbb{S}^1), \quad \gamma^+(\theta) > \gamma^-(\theta) \text{ for all } \theta\}.$$

The restriction on existence above suggests that we focus on the open subset

$$\mathfrak{C}^\pi = \{(\gamma^+, \gamma^-) \in \mathfrak{C} : \sup_{\theta \in \mathbb{S}^1} (\gamma^+(\theta) - \gamma^-(\theta)) < \pi\}.$$

We also define \mathfrak{A} to be the space of properly embedded minimal annuli with $\partial A \in \mathfrak{C}$. Our main results will be phrased in terms of properties of the natural projection map

$$\Pi : \mathfrak{A} \longrightarrow \mathfrak{C}, \quad \Pi(A) := \partial A.$$

The first result is the easiest one to state. Consider the subspaces $\mathfrak{C}_m \subset \mathfrak{C}^\pi$ and \mathfrak{A}_m of boundary curves and minimal annuli which are invariant under the discrete group of isometries generated by the rotation R_m by angle $2\pi/m$ about the axis $\{o\} \times \mathbb{R}$. Imposing symmetry eliminates a degeneracy in the problem.

Theorem 1.1. *For any $m \geq 2$,*

$$\Pi|_{\mathfrak{A}_m} : \mathfrak{A}_m \rightarrow \mathfrak{C}_m$$

is surjective.

It is not the case that the full map $\Pi : \mathfrak{A} \rightarrow \mathfrak{C}^\pi$ is surjective, and indeed, we present below a simple and large family of examples of pairs of curves which do not bound minimal annuli. Thus we prove a slightly weaker existence result.

Theorem 1.2. *Given any $(\gamma^+, \gamma^-) \in \mathcal{C}^{2,\alpha}(\mathbb{S}^1)^2$, there exist constants a_0, a_1, a_2 so that the pair $(\gamma^+ + a_0 + a_1 \cos \theta + a_2 \sin \theta, \gamma^-)$ bounds a properly Alexandrov-embedded, minimal annulus.*

Remark 1.3. There is a very important difference between this result and how we have tried to formulate the result previously. First, we are not specifying the boundary curves completely, but allowing a three-dimensional freedom in the top curve. Second, and of fundamental importance, we pass from the space of properly embedded to (properly) Alexandrov-embedded minimal annuli. We denote this space by \mathfrak{A}^* . It is most likely impossible to characterize the precise set of pairs of boundary curves for which the minimal annuli provided by this theorem are actually embedded, but if we allow Alexandrov-embeddedness, there is a satisfactory global existence theorem. For the subclasses \mathfrak{A}_m and \mathfrak{C}_m however, it is possible to remain within the class of embedded surfaces.

The strategy to prove both of these theorems uses degree theory in a familiar way. The main step is to show that Π is a proper Fredholm map. This is true for the restriction of Π to \mathfrak{A}_m , but unfortunately may not be the case on all of \mathfrak{A} , so instead we consider a finite dimensional extension of Π which is proper, but which leads to the need to introduce the extra flexibility in the top boundary curve.

After setting forth some notation and basic analytic and geometric facts in the next section, we present the nonexistence examples in §3, followed by the calculation of fluxes on horizontal ends in §4. Next, §5 contains an extension of a theorem of Colin, Hauswirth and Rosenberg [1] and proves that the ends of elements of \mathfrak{A} are indeed vertical graphs.

Proposition 1.4. *If $A \in \mathfrak{A}$, then there is a compact set $K \subset A$ such that $A \setminus K = E^+ \sqcup E^-$, where each E^\pm is a vertical graph of a function u^\pm over some region $\{r_0 < |z| < 1\}$.*

By an observation in [5], u^\pm extends to a $\mathcal{C}^{2,\alpha}$ function up to $|z| = 1$, or equivalently, \bar{A} is a $\mathcal{C}^{2,\alpha}$ surface with boundary. We prove in §6 that the space \mathfrak{A} is a Banach manifold and study the space of Jacobi fields on a minimal annulus A . This leads to the definition, in §7, of the extended boundary map $\widetilde{\Pi}$, and an exploration of its infinitesimal properties. The more difficult fact that $\widetilde{\Pi}$ is proper occupies §8. We finally prove the two main theorems in §9 and §10.

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2. VERTICAL CATENOIDS

In this section we recall the salient geometric and analytic properties of the vertical minimal catenoids.

As in the introduction, we use the Poincaré disk model for \mathbb{H}^2 , with Cartesian coordinates $z \in D(o, 1) = \{z : |z| < 1\}$ and polar coordinates (r, θ) . We shall also use the notation $D(z_0, R)$ and $D_{\mathbb{H}^2}(z_0, R)$ to denote the Euclidean and hyperbolic disks with center z_0 and (Euclidean or hyperbolic) radius R . When $z_0 = 0$, we sometimes omit it from the notation.

2.1. Geometric properties. The family of vertical minimal catenoids was introduced and studied by Nelli and Rosenberg [9] as the unique family of minimal surfaces invariant under rotations around a fixed vertical axis. Indeed, parametrizing a surface of rotation by

$$[0, 2\pi] \times (a, b) \ni (\theta, t) \mapsto X(\theta, t) := (r(t)e^{i\theta}, t),$$

then minimality is equivalent to the equation

$$4rr'' - 4(r')^2 - (1 - r^4) = 0.$$

Integrating this gives that

$$(2.1) \quad \kappa^2 = \frac{(1 - r^2)^2}{4r^2} + \left(\frac{r'}{r}\right)^2$$

for some constant $\kappa^2 > 0$. It can then be deduced that solutions exist on some interval (a, b) with $b - a = 2h < \pi$, and furthermore that the correspondence $(0, \pi/2) \ni h \rightarrow \kappa^2 \in \mathbb{R}^+$ is bijective. We denote the corresponding surface by

C_h , usually with the normalization that $a = -h$, $b = h$, hence $r(-h) = r(h) = 1$. Note that the first-order equation for r implies that

$$\sqrt{1 + \kappa^2} - \kappa \leq r(t) < 1,$$

and this lower bound is the minimum of the corresponding solution $r(t)$; this provides a correspondence between κ and the minimum value $r(0)$.

We now calculate that

$$X^*(g|_{C_h}) = \frac{4r^2}{(1-r^2)^2}(d\theta^2 + \kappa^2 dt^2).$$

This is nearly conformal and the extra constant factor (which could obviously be scaled away) does not cause any problems.

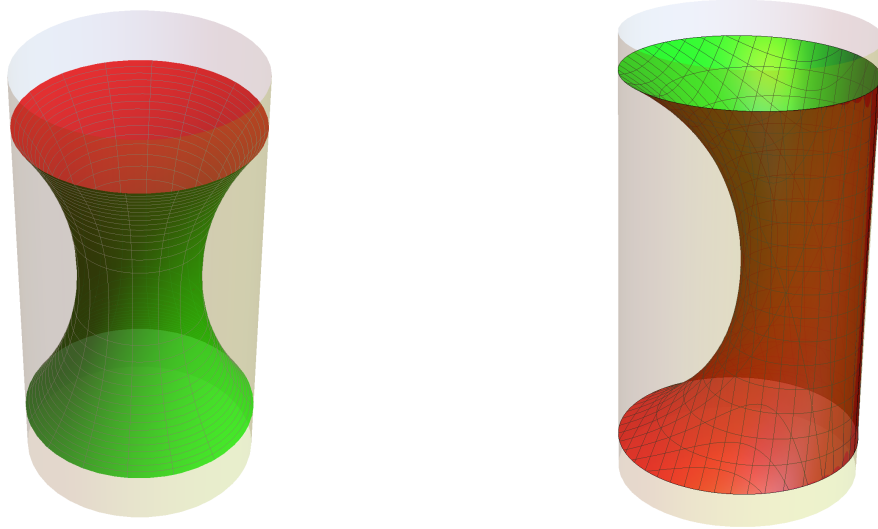


FIGURE 1. Vertical catenoid and parabolic generalized catenoid.

These surfaces have the following properties:

- (1) C_h is a bigraph with respect to the horizontal plane $\mathbb{H}^2 \times \{0\}$.
- (2) As $h \searrow 0$, C_h converges to $\mathbb{H}^2 \times \{0\}$, branched at the origin, with multiplicity 2.
- (3) As $h \nearrow \pi/2$, C_h diverges to the cylindrical wall at infinity $\partial\mathbb{H}^2 \times (-\pi/2, \pi/2)$.

Remark 2.1. Let T_{z_0} denote the horizontal dilation that maps $(0, t)$ into $(z_0, t) \in \mathbb{H}^2 \times \mathbb{R}$,

$$T_{z_0}(z, t) := \left(\frac{z + z_0}{\bar{z}_0 z + 1}, t \right),$$

and define $C_{h,z_0} := T_{z_0}(C_h)$. When $z_0 = 0$ we simply write C_h . Then the family $\mathcal{M} := \{C_{h,z_0} : (z_0, h) \in \mathbb{H}^2 \times \mathbb{R}\}$ forms a 3-dimensional submanifold of the Banach manifold of all annuli.

2.2. Parabolic generalized catenoids. Although C_h diverges as $h \rightarrow \pi/2$, one can obtain a nontrivial limit of this family as follows. For each h apply a hyperbolic isometry T_h of \mathbb{H}^2 , acting trivially on the \mathbb{R} factor, which translates along a fixed geodesic passing through the origin. We fix T_h completely by demanding that $(0, 0) \in T_h(C_h)$; the tangent plane at that point is then necessarily vertical. There exists a nontrivial limit $\mathcal{D} = \mathcal{D}_q$ of the $T_h(C_h)$, discovered originally by Daniel [3, 4]. Its asymptotic boundary consists of the two circles $\mathbb{S}^1 \times \{\pm\pi/2\}$ together with a vertical segment $\{q\} \times [-\pi/2, \pi/2]$. It is foliated by horocycles $H_t = \mathcal{D} \cap (\mathbb{H}^2 \times \{t\})$ based at the point q . Note that applying other horizontal dilations along the same geodesic produces a family of minimal surfaces with the same asymptotic boundary which foliate the slab $\mathbb{H}^2 \times (-\pi/2, \pi/2)$; one limit of this family is the two disks $\mathbb{H}^2 \times \{\pm\pi/2\}$. We shall refer them as *parabolic generalized catenoids*.

These families of surfaces enjoy the following uniqueness properties:

- (1) (Nelli, Sa Earp & Toubiana [10]): A minimal annulus bounded by $(\mathbb{S}^1 \times \{\pm h\})$, for any $h \in (0, \pi/2)$, must equal C_{h,z_0} for some $z_0 \in \mathbb{H}^2$.
- (2) (Daniel, Hauswirth [3, 4]): If X is any (possibly incomplete) minimal surface in $\mathbb{H}^2 \times \mathbb{R}$, the intersection of which with any horizontal plane $\mathbb{H}^2 \times \{t\}$ is a piece of a horocycle, then it must be a piece of some parabolic generalized catenoid.

These are interesting model examples and provide very useful barriers, see §3 for example.

2.3. The Jacobi operator. We now derive an explicit expression for the Jacobi operator L on C_h and determine the space of decaying Jacobi fields.

We first recall the Jacobi operator

$$L_C = \Delta_C + |S_C|^2 + \text{Ric}(\nu, \nu),$$

where S_C is the shape operator (or second fundamental form) of C . Rather than computing the last two terms explicitly, we use the coordinates and explicit form of the metric above to note that

$$L = \frac{(1-r^2)^2}{4r^2}(\kappa^{-2}\partial_t^2 + \partial_\theta^2 + q(t)),$$

for some function q . We can determine q by plugging in a known solution of $Lw = 0$, and we shall use the Jacobi field w arising from vertical translations.

To calculate this Jacobi field, we first compute the unit normal to C ,

$$\nu = \left(\frac{(1-r^2)^2}{4\kappa r} \cos \theta, \frac{(1-r^2)^2}{4\kappa r} \sin \theta, -\frac{r'}{\kappa r} \right).$$

The Killing field generated by vertical translation is $(0, 0, 1)$, hence its projection onto ν is simply $-1/\kappa$ times the function r'/r . In other words, $L_C(r'/r) = 0$. A short calculation then gives that

$$q = \frac{1}{2\kappa^2}(r^{-2} + r^2),$$

so altogether,

$$(2.2) \quad L_C = \frac{(1-r^2)^2}{4\kappa^2 r^2} \left(\partial_t^2 + \kappa^2 \partial_\theta^2 + \frac{1}{2} \left(\frac{1}{r^2} + r^2 \right) \right).$$

Now set

$$\mathfrak{J}(C) = \{\psi \in L^\infty(C) : L_C \psi = 0\}, \quad \mathfrak{J}^0(C) = \mathfrak{J}(C) \cap L^2(C);$$

we shall actually consider only the subclass of Jacobi fields which extend to be $\mathcal{C}^{2,\alpha}$ on \overline{C} , but this will be discussed in detail in §6, along with many further properties of the Jacobi operator, both at C and at any other $A \in \mathfrak{A}$. The space $\mathfrak{J}(C)$ is infinite dimensional and is (almost) parametrized by its asymptotic boundary values. We note one special fact that if $\phi \in \mathfrak{J}^0(C)$, then ϕ is automatically smooth up to ∂C as a function of (r, θ) , and the L^2 condition means that it vanishes like $1-r$.

Expanding any function u on C as

$$u(t, \theta) = \sum_{n=0}^{\infty} (\alpha_n(t) \cos(n\theta) + \beta_n(t) \sin(n\theta)), \quad (\text{with } \beta_0 \equiv 0),$$

then

$$L_C u = \sum_{n=0}^{\infty} ((L_n \alpha_n) \cos(n\theta) + (L_n \beta_n) \sin(n\theta)),$$

where

$$L_n = \frac{(1-r^2)^2}{4\kappa^2 r^2} \left(\partial_t^2 - \kappa^2 n^2 + \frac{1}{2} \left(\frac{1}{r^2} + r^2 \right) \right).$$

Proposition 2.2. *The space of decaying Jacobi fields $\mathfrak{J}^0(C_h)$ is spanned by $\varphi_1 = \phi(r) \cos \theta$ and $\varphi_2 = \phi(r) \sin \theta$, where*

$$\phi(r) := \frac{1}{r} - r.$$

These are the Jacobi fields generated by horizontal dilations.

Proof. We must determine all solutions to $L_n u = 0$ with $u(\pm h) = 0$. First observe that $L_1 \phi = 0$ where ϕ is given in the statement of the theorem. The Sturm-Picone comparison theorem then gives that any solution of the Dirichlet problem for L_n , $n \geq 1$, must be proportional to ϕ , which is impossible for $n \geq 2$.

There is at most a two dimensional space of solutions to $L_0 u = 0$, and a basis for this space is given by the Jacobi fields generated by vertical translations and by varying the parameter h . We have already computed the first of these, which is the function r'/r , which does not vanish at $\pm h$. It is easy to see that the second one cannot vanish at $\pm h$ either. \square

3. NONEXISTENCE

We present here two separate results which limit the types of pairs of curves which can arise as boundaries of minimal annuli. The first proof is based on a standard barrier argument and the second on Alexandrov reflection.

Proposition 3.1. *Suppose that $\Gamma = (\gamma^+, \gamma^-) \in \mathfrak{C}$, but that $\gamma^+(\theta) \geq \pi$ and $\gamma^-(\theta) \leq 0$ for every θ (with at least one inequality strict at some point). Then there exists no complete connected properly embedded minimal surface A with horizontal ends and with $\partial A = \Gamma$.*

Proof. Fix any point $q \in \partial \mathbb{H}^2$ and consider the family of parabolic generalized catenoids $\mathcal{D}_{q,s}$ with fixed vertical boundary $\{q\} \times [0, \pi]$, where s is the horizontal dilation parameter. Since $\mathcal{D}_{q,s}$ is disjoint from every set of the form $K \times [\epsilon, \pi - \epsilon]$ when $s \gg 1$, say, where $K \subset \mathbb{H}^2$ is compact, we can decrease s until the point of first contact, which is impossible. \square

Remark 3.2. It is very likely that the correct statement is that there is no connected minimal surface with boundary Γ for any pair of boundary curves for which $\gamma^+(\theta) \geq \gamma^-(\theta) + \pi$ at every θ . We could prove this in the same way if there were to exist a sufficiently rich family of ‘deformed’ parabolic generalized catenoids. We hope to return to this elsewhere.

For the next result we impose a monotonicity condition on $\Gamma = (\gamma^+, \gamma^-)$, which we normalize by centering around $\theta = 0, \pi$. Thus suppose that γ^- is monotone decreasing on $[0, \pi]$ and monotone increasing on $[\pi, 2\pi]$, while γ^+ is monotone increasing on $[0, \pi]$ and monotone decreasing on $[\pi, 2\pi]$. In other words, the two curves are tilted away from each other. We allow non-strict monotonicity in each interval, but exclude the one case where both γ^+ and γ^- are at a constant height.

Proposition 3.3. *Under the conditions above, there is no $A \in \mathfrak{A}$ with $\partial A = \Gamma$ unless γ^\pm are constant (in which case A is a catenoid).*

Proof. Let $\eta(s)$ denote the geodesic in \mathbb{H}^2 which connects $(1, 0)$ (where $\theta = 0$) to $(-1, 0)$, with $\eta(0) = (0, 0)$, with $\eta \rightarrow (1, 0)$ as $s \rightarrow -\infty$. For each s denote by λ_s the geodesic orthogonal to η and meeting it at $\eta(s)$. The vertical plane $P_s = \lambda_s \times \mathbb{R}$ separates $\mathbb{H}^2 \times \mathbb{R}$ into two components, \mathcal{U}_s and \mathcal{V}_s , and we assume that $(1, 0, 0) \in \overline{\mathcal{U}_s}$.

First assume that both inequalities $\gamma^+(\pi) > \gamma^+(0)$ and $\gamma^-(\pi) < \gamma^-(0)$ are strict. Write $A'_s = A \cap \overline{\mathcal{U}_s}$, $A''_s = A \cap \overline{\mathcal{V}_s}$, and denote by A_s^* the reflection of A'_s into $\overline{\mathcal{V}_s}$. We prove in §5 that each end of A is a vertical graph, $\mathbb{H}^2 \setminus B(o, R) \ni z \mapsto (z, u^\pm(z))$. Since $u^\pm(1, \theta) = \gamma^\pm(\theta)$, the monotonicity hypotheses imply that when $s \ll 0$, the boundary curves $(\gamma_s^*)^\pm$ of A_s^* satisfy

$$\gamma^-(\theta) \leq (\gamma_s^*)^-(\theta) < (\gamma_s^*)^+(\theta) \leq \gamma^+(\theta)$$

for all θ with $e^{i\theta} \times \mathbb{R} \subset \overline{\mathcal{V}_s}$. In addition, $\partial A_s^* \cap P_s = A \cap P_s$. Certainly A_s^* does not make contact with A''_s except at the boundary when s is very negative. We then let s increase until the first point of interior contact, which shows that $A_s^* = A''_s$ for some s . However, by the monotonicity hypotheses, this is impossible.

The one case left to analyze is when one side, say γ^- is constant while $\gamma^+(\pi) > \gamma^+(0)$. We can proceed exactly as before provided we can begin the reflection argument. For this, it suffices to show that A_s^* lies strictly above A''_s when $s \ll 0$. In this case, these two surfaces have the same bottom boundary. The concern is that A_s^* might dip below A''_s . To rule this out we consider the family of vertical translates $A_s^* + (0, 0, \tau)$, $\tau < 0$. By letting $\tau \nearrow 0$ we would attain an interior point of tangential contact, which is impossible. Hence we may proceed exactly as before. The conclusion remains that there is no such minimal annulus A with this type of boundary. \square

Remark 3.4. The results of this section are still true if the annulus A is Alexandrov-embedded.

4. FLUXES OF MINIMAL ANNULI

Let A be a minimal surface lying in an ambient space which has a continuous families of isometries, and γ any closed curve in A . For any Killing field Z on the ambient space, the flux of A across γ , $\text{Flux}(A, \gamma, Z)$, depends only on the homology class of γ in A . This flux is defined by integrating $\langle Z, \eta \rangle$ around γ , where η is the unit normal to γ in A . We now compute these flux integrals for minimal annuli in $\mathbb{H}^2 \times \mathbb{R}$ with horizontal ends, where γ is the generating loop for the homology $H^1(A, \mathbb{Z})$. These invariants play an important role later in this paper.

Fix $A \in \mathfrak{A}$, $\Gamma = (\gamma^+, \gamma^-) = \partial A$. As proved in §5, each end of A is a vertical graph over some region $\{z : R \leq |z| < 1\}$, with graph functions u^\pm , so we use the graphical representations

$$X^\pm(r, \theta) = (r \cos \theta, r \sin \theta, u^\pm(r, \theta)), \quad R \leq r < 1, \quad \theta \in \mathbb{S}^1.$$

Consider the restriction of X^\pm to $X^\pm \cap \{r = \text{const.}\}$ and the orthonormal frame

$$E_1 = (\sqrt{F}, 0, 0), \quad E_2 = (0, \sqrt{F}, 0), \quad E_3 = (0, 0, 1),$$

where $F = \frac{1}{4}(1 - r^2)^2$. Evaluating all functions at this fixed value of r ,

$$(\gamma^\pm)'(r, \theta) = -\frac{r \sin \theta}{\sqrt{F}} E_1 + \frac{r \cos \theta}{\sqrt{F}} E_2 + u_\theta^\pm E_3,$$

hence the unit tangent to $\theta \mapsto \gamma^\pm$ is

$$T^\pm(\theta) = \frac{\gamma_\theta^\pm}{\|\gamma_\theta^\pm\|_g}, \quad \|\gamma_\theta^\pm\|_g = \frac{\sqrt{r^2 + F(u_\theta^\pm)^2}}{\sqrt{F}}.$$

Similarly, the normal to this curve in A equals

$$\eta^\pm(\theta) = \pm Q \left(\left(\frac{r \cos \theta}{\sqrt{F}} + \sqrt{F} u_\theta^\pm u_{x_2}^\pm \right) E_1 + \left(\frac{r \sin \theta}{\sqrt{F}} - \sqrt{F} u_\theta^\pm u_{x_1}^\pm \right) E_2 + r u_r^\pm E_3 \right),$$

where

$$Q = \left(\|\gamma_\theta^\pm\|_g \sqrt{1 + F \|\nabla u^\pm\|^2} \right)^{-1}.$$

We now compute the fluxes with respect to E_3 and the horizontal Killing fields generated by rotations and hyperbolic dilations.

The first is the simplest:

$$\begin{aligned} \text{Flux}(A, \gamma^\pm, E_3) &= \int_{\gamma^\pm} \langle \eta^\pm, E_3 \rangle d\sigma = \\ &= \pm \int_0^{2\pi} \frac{r u_r^\pm}{\sqrt{1 + F \|\nabla u^\pm\|^2}} d\theta = \pm \int_0^{2\pi} \frac{r u_r^\pm}{\sqrt{1 + \frac{1}{4}(1 - r^2)^2 ((u_r^\pm)^2 + r^{-2}(u_\theta^\pm)^2)}} d\theta. \end{aligned}$$

This is constant in r , and the limit as $r \nearrow 1$ equals

$$\text{Flux}(A, \gamma^\pm, E_3) = \pm \int_0^{2\pi} u_r^\pm(1, \theta) d\theta.$$

Next, the Killing field generated by rotations around the vertical axis $\{0\} \times \mathbb{R}$ is

$$Z = -\frac{r \sin \theta}{\sqrt{F}} E_1 + \frac{r \cos \theta}{\sqrt{F}} E_2,$$

and we have

$$\text{Flux}(A, \gamma^\pm, Z) = \mp \int_0^{2\pi} \frac{r u_\theta^\pm u_r^\pm}{\sqrt{1 + F \|\nabla u^\pm\|^2}} d\theta.$$

Letting $r \nearrow 1$ as before gives that

$$(4.1) \quad \text{Flux}(A, \gamma^\pm, Z) = \mp \int_0^{2\pi} u_r^\pm(1, \theta) u_\theta^\pm(1, \theta) d\theta.$$

Finally, consider the Killing fields

$$H_a = \frac{r^2 \cos(a - 2\theta) - \cos(a)}{2\sqrt{F}} E_1 - \frac{r^2 \sin(a - 2\theta) + \sin(a)}{2\sqrt{F}} E_2,$$

$a \in [0, 2\pi)$, corresponding to the horizontal dilations along the geodesic joining e^{ia} and -1 . We calculate

$$\begin{aligned} \text{Flux}(A, \gamma^\pm, H_a) &= \int_{\gamma^\pm} \langle \eta^\pm, H_a \rangle d\sigma \\ &= \pm \int_0^{2\pi} \frac{1}{2\sqrt{1 + F\|\nabla u^\pm\|^2}} \left(\left(\frac{4r}{r^2 - 1} + \frac{(r^2 - 1)(u_\theta^\pm)^2}{r} \right) \cos(a - \theta) \right. \\ &\quad \left. + (r^2 + 1)u_r^\pm u_\theta^\pm \sin(a - \theta) \right) d\theta. \end{aligned}$$

Unlike the previous cases we cannot take limits directly since the first term appears to diverge. However,

$$\frac{1}{2\sqrt{1 + F\|\nabla u^\pm\|^2}} \left(\frac{4r}{r^2 - 1} + \frac{(r^2 - 1)(u_\theta^\pm)^2}{r} \right) = \frac{1}{r - 1} + \frac{1}{2} + \mathcal{O}(1 - r).$$

When we multiply by $\cos(\theta - a)$ and integrate in θ , the first two terms on the right vanish, while the third vanishes in the limit as $r \nearrow 1$. Therefore only the final term remains and we obtain that

$$\text{Flux}(A, \gamma^\pm, H_a) = \pm \int_0^{2\pi} u_r^\pm(1, \theta) u_\theta^\pm(1, \theta) \sin(\theta - a) d\theta.$$

These computations prove the following

Lemma 4.1. *Let A be a complete properly embedded minimal annulus with horizontal ends, parametrized as above. Then*

$$(4.2) \quad \int_0^{2\pi} u_r^+(1, \theta) d\theta + \int_0^{2\pi} u_r^-(1, \theta) d\theta = 0,$$

$$(4.3) \quad \int_0^{2\pi} u_r^+(1, \theta) u_\theta^+(1, \theta) d\theta + \int_0^{2\pi} u_r^-(1, \theta) u_\theta^-(1, \theta) d\theta = 0,$$

and for every $a \in [0, 2\pi)$,

$$(4.4) \quad \int_0^{2\pi} u_r^+(1, \theta) u_\theta^+(1, \theta) \sin(\theta - a) d\theta + \int_0^{2\pi} u_r^-(1, \theta) u_\theta^-(1, \theta) \sin(\theta - a) d\theta = 0,$$

5. GRAPHICAL PARAMETRIZATION OF HORIZONTAL ENDS

In this section we extend and sharpen a result of Collin, Hauswirth and Rosenberg [1] and prove that any properly Alexandrov-embedded, minimal annulus A can be written as a vertical graph near infinity.

In the argument below we use the family of ‘tall rectangles’ obtained in [3, 4, 12] (see also [7]), and we begin by recalling these. Let σ be any connected arc in $\mathbb{S}^1 = \partial\mathbb{H}^2$ and $a, b \in \mathbb{R}$ with $b - a > \pi$. Suppose also that η is the geodesic in \mathbb{H}^2 with the same endpoints as σ . Then there is a minimal disk $R(\sigma, a, b)$ with asymptotic boundary equal to the rectangle $(\sigma \times \{a, b\}) \cup (\partial\sigma \times [a, b])$, and such that each intersection $R(\sigma, a, b) \cap (\mathbb{H}^2 \times \{t_0\})$ projects on \mathbb{H}^2 to a curve equidistant from the geodesic η . The distance of the ‘central’ slice (at height $(a + b)/2$) to η tends to 0 as $b - a \nearrow \infty$ so it makes sense to define $R(\sigma, -\infty, \infty)$ to be the vertical plane $\eta \times \mathbb{R}$; there are also semi-infinite tall rectangles $R(\sigma, -\infty, b)$ and $R(\sigma, a, \infty)$. The distance from the central slice to η tends to infinity as $b - a \searrow \pi$, and in fact if we simultaneously let σ converges to the entire circle and $b - a \searrow \pi$ then $R(\sigma, a, b)$ converges to a parabolic generalized catenoid, which is foliated by horocycles (see Section 2.2).

The other tool in this proof is the so-called ‘Dragging Lemma’ (which we state in a slightly simplified version adapted to our purposes):

Lemma 5.1 ([1]). *Let E be a properly embedded annular end in $\mathbb{H}^2 \times \mathbb{R}$ so that one component of its boundary, ∂E , is a closed loop in the interior of this space and its virtual boundary at infinity, $\partial_\infty E = \gamma$, is a vertical graph of a continuous function $\gamma(\theta)$ in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$. Suppose that Y_s is a one-parameter family of compact minimal surfaces with boundary in $\mathbb{H}^2 \times \mathbb{R}$ such that $Y_s \cap \partial E = \emptyset$ and $\partial Y_s \cap E = \emptyset$ for each s . Suppose that $p_0 \in Y_0 \cap E$. Then there exists a continuous curve $s \mapsto p_s$ such that $p_s \in E \cap Y_s$ for each s which equals p_0 at $s = 0$.*

Now we turn to the main result of this section:

Proposition 5.2. *Let E be a properly embedded annular end as in the previous lemma. Then for a sufficiently large $R > 0$, there exists a function $u : \mathbb{H}^2 \setminus B_R(o) \rightarrow \mathbb{R}$ with $u(1, \theta) = \gamma(\theta)$ such that the graph of u equals $E_R := E \cap ((\mathbb{H}^2 \setminus B_R(o)) \times \mathbb{R})$.*

Remark 5.3. The graph function u is smooth in the interior and as regular at $\partial_\infty E$ as the function $\gamma(\theta)$.

Proof. For each $\epsilon > 0$, there exists $\delta > 0$ so that if σ is an arc in \mathbb{S}^1 of length less than δ , then $\sup_\sigma \gamma - \inf_\sigma \gamma < \epsilon$. Now consider the semi-infinite tall rectangles $R(\sigma, -\infty, \inf_\sigma \gamma)$ and $R(\sigma, \sup_\sigma \gamma, \infty)$. If σ is small enough, these do not intersect ∂E , and by the maximum principle, neither intersects E in its interior.

Fixing a large constant $C_1 < \frac{\pi}{4\epsilon}$, there exists a curve $\tilde{\eta}$ equidistant from the geodesic η associated to σ so that in the lens-shaped region D_σ between σ and $\tilde{\eta}$ the vertical distance between these upper and lower semi-infinite tall rectangles is less than $C_1\epsilon$ (see Fig. 2.) Fixing ϵ and C_1 , we can cover \mathbb{S}^1 by finitely many such arcs σ ; the union of the corresponding lenses D_σ covers an outer annular region $D = \mathbb{H}^2 \setminus D(o, R_0)$, in which the difference between the maximum and minimum

height of $E \cap (\{p\} \times \mathbb{R})$ is less than $C_1\epsilon$. Clearly $E \cap (D \times \mathbb{R})$ is trapped in the region between the union of the upper and of the lower semi-infinite tall rectangles.

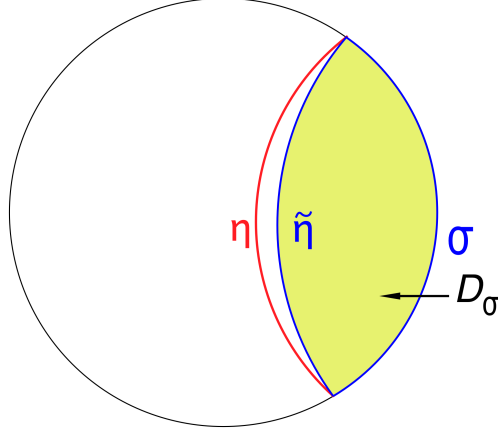


FIGURE 2. The region D_σ .

Next fix a truncated vertical catenoid $C_{h,\rho}$, i.e., the intersection of the catenoid C_h centered on the axis $\{o\} \times \mathbb{R}$ of height h with the cylinder $B(o, \rho) \times \mathbb{R}$. We set $h = 4C_1\epsilon$, where C_1 and ϵ are as above, and choose ρ sufficiently large so that the vertical separation between the upper and lower boundaries of $C_{h,\rho}$ is greater than $3C_1\epsilon$. Suppressing h and ρ from the notation, denote by $C(q, \tau)$ the translate of this truncated catenoid by isometries of $\mathbb{H}^2 \times \mathbb{R}$ so that it is centered at $(q, \tau) \in \mathbb{H}^2 \times \mathbb{R}$. By the construction above, we may choose a radius $R_1 \gg R_0$ so that $\partial E \subset B(o, R_1) \times \mathbb{R}$, and a continuous function $q \mapsto \tau(q)$, $q \in \mathbb{H}^2 \setminus B(o, R_1)$, satisfying that

$$(5.1) \quad C(q, \tau(q)) \cap \partial E = \emptyset, \text{ and } \partial C(q, \tau(q)) \cap E = \emptyset$$

for every q in this exterior region.

Let us define $E_R := E \cap (B(o, R) \times \mathbb{R})$. To prove the theorem, we must verify the following two assertions for $R \gg R_1$:

- i) the projection $E_R \rightarrow \mathbb{H}^2 \setminus B(o, R)$ is bijective and
- ii) there are no points $(q, t) \in E_R$ such that $T_{(q,t)}E$ is vertical.

First note that i) is a consequence of ii). Indeed, if the projection of E_R does not contain a full neighborhood of infinity, then there exists a sequence of points $q_j \in \mathbb{H}^2$ which tend to infinity and which do not lie in this image. Because E is properly embedded, its projection on \mathbb{H}^2 is closed, so for each j there exists $\kappa_j > 0$ such that $B(q_j, \kappa_j)$ is also disjoint from this image. Next, by translating the center q_j further out from o in the component of the complement of the image of the projection of

E , we may arrange that $\partial B(q_j, \kappa_j) \times \mathbb{R}$ is tangent to E at some point (q'_j, t_j) , and clearly the tangent plane of E must be vertical there. This proves the surjectivity of this projection. Furthermore, if ii) holds, then the projection is a covering map, and properness of the embedding prevents there from being more than one sheet. Hence it must be bijective, and therefore a diffeomorphism.

We therefore turn to assertion ii). Suppose, to the contrary, that there is a sequence of points (q_j, t_j) diverging in $\mathbb{H}^2 \times \mathbb{R}$ and a sequence of geodesics η_j such that the vertical plane $P_j = \eta_j \times \mathbb{R}$ is tangent to E at (q_j, t_j) . For each j consider the intersection $N_j = P_j \cap E$. This is the intersection of two minimal surfaces, and hence locally homeomorphic to the zero set of a harmonic function. By the maximum principle, it contains no isolated points, and in some neighborhood of (q_j, t_j) , N_j is the union of $k \geq 2$ smooth curves which meet there at equally spaced angles. In other words, N_j is a network of curves in P_j .

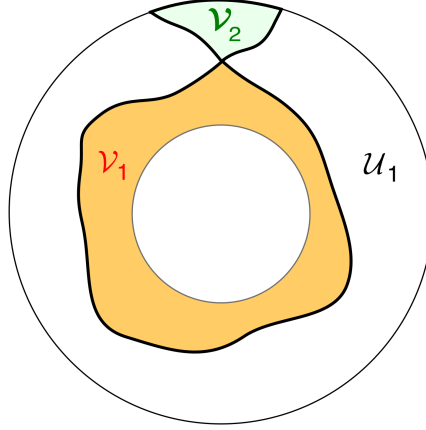
Let us begin by considering the case where $\eta_j \times \mathbb{R}$ is disjoint from $B(o, R_1 + \rho) \times \mathbb{R}$ (so in particular $P_j \cap \partial E = \emptyset$), and describe some global properties of this network in that case. First, as just noted, $\text{dist}(N_j, \partial E)$ can be assumed to be arbitrarily large. Next, N_j intersects the outer boundary of E in at most two points: p_∞^1, p_∞^2 . Now, the complement $E \setminus N_j$ is a disjoint union of two collections of connected open sets, $\mathcal{U}_1 \cup \mathcal{U}_2 \cup \dots$ and $\mathcal{V}_1 \cup \mathcal{V}_2 \cup \dots$, where each \mathcal{U}_i lies in the component of $(\mathbb{H}^2 \times \mathbb{R}) \setminus P_j$ which does not contain ∂E , while each \mathcal{V}_i lies on the side of P_j containing ∂E . Precisely one of the \mathcal{V}_i , say \mathcal{V}_1 , contains ∂E .

It is impossible that any \mathcal{U}_i or any \mathcal{V}_i except \mathcal{V}_1 has compact closure. Indeed, if that were the case, then the closure of that component would be a compact minimal surface with boundary in a vertical plane, which is impossible by the maximum principle (the complement of that vertical plane is foliated by a family of vertical planes). We also claim that it is impossible that the closure of any \mathcal{V}_i , $i \neq 1$, or \mathcal{U}_i , contains only a single point at infinity, i.e., either p_∞^1 or p_∞^2 , but not a full arc of the boundary curve. The reason is the same: the corresponding truncated minimal surface would have boundary contained in the closure of P_j , and one could find a disjoint vertical plane which touches it at only one point.

There is only one alternative remaining, which is that there are precisely two curves in N_j intersecting at (q_j, t_j) , and that there is a single component \mathcal{U}_1 on the ‘outside’ of P_j and two components \mathcal{V}_1 and \mathcal{V}_2 on the same side of P_j as ∂E (see Fig. 3.)

Now consider the component \mathcal{V}_1 , whose boundary components are: ∂E and the closed loop $\partial \mathcal{V}_1 \subset P_j$ passing through (q_j, t_j) .

Let η_ρ denote the curve equidistant from η at distance ρ and contained in the same component of $\mathbb{H}^2 - \eta_j$ as ∂E . Since the region between η and η_ρ is included in $\mathbb{H}^2 \setminus B(o, R_1)$ we have from (5.1) that the translated catenoids $C(q_s, \tau(q_s))$, where


 FIGURE 3. The connected components of $A \setminus P_j$.

q_s is a parametrization of the curve η_ρ , satisfy

$$C(q_s, \tau(q_s)) \cap \partial E = \emptyset \quad \text{and} \quad \partial C(q_s, \tau(q_s)) \cap E = \emptyset.$$

Hence by the Dragging Lemma (Lemma 5.1) there exists a continuous curve $s \mapsto p_s$ such that $p_s \in C(q_s, \tau(q_s)) \cap E$ for each s such that $p_0 \in \mathcal{V}_1$ and $p_s \rightarrow \infty$ as $s \rightarrow \infty$. Since $p_s \notin P_j$ for any s we conclude that the entire curve is in \mathcal{V}_1 . This too is impossible as \mathcal{V}_1 has compact closure.

It remains to consider the case where P_j intersects $B(o, R_1 + \rho) \times \mathbb{R}$ for all but finitely many j . We shall show that the vertical plane P_j can be replaced by a tall rectangle of finite height, $R_j = R(\sigma_j, a_j, b_j)$, which is tangent to E (and P_j) at (q_j, t_j) , and which does not meet $B(o, R_1 + \rho) \times \mathbb{R}$. To demonstrate this, first perform an isometry T_j which moves (q_j, t_j) to $(o, 0)$ and so that the horizontal geodesic η_j is transformed to a fixed geodesic η . The transformed ball $B(Q_j, R_1) = T_j(B(o, R_1))$ must meet η , and does so at a distance from $o \in \mathbb{H}^2$ which tends to infinity with j . Furthermore, since R_1 is fixed, the geodesic cone emanating from o and subtending $B(Q_j, R_1)$ has angle ω_j which tends to 0 as $j \rightarrow \infty$.

Fix $\delta' > 0$ and consider the geodesic $\hat{\eta}$ in \mathbb{H}^2 at distance precisely δ' from η such that o is the closest point on η to $\hat{\eta}$ as well as the arc of the boundary circle $\hat{\sigma}$ with the same endpoints as $\hat{\eta}$ and on the same side of $\hat{\eta}$ as η . There is a unique tall rectangle $R(\hat{\sigma}, -h', h')$ which meets $\mathbb{H}^2 \times \{0\}$ at the curve equidistant from $\hat{\eta}$ at distance precisely δ' . This tall rectangle is tangent to $\eta \times \mathbb{R}$ at $(o, 0)$, and hence to $T_j(E)$ as well. Note that δ' and h' are determined in terms of one another, and $h' \rightarrow \infty$ as $\delta' \rightarrow 0$. We claim that when h' is fixed sufficiently large and j is also large enough, this rectangle does not intersect the cylindrical slab $B(Q_j, R_1) \times [-C_2, C_2]$,

where C_2 is chosen so that $T_j(E) \subset \mathbb{H}^2 \times [-C_2, C_2]$ for all j . Indeed, fix $h' > 3C_2$ and observe that the lunette region Ω between the core curve $R(\hat{\sigma}, -h', h') \cap \{t = 0\}$ and the equidistant curve which is the projection to \mathbb{H}^2 of $R(\hat{\sigma}, -h', h') \cap \{t = \pm 2C_2\}$ meets $\partial\mathbb{H}^2$ at the two endpoints of $\hat{\eta}$ and its intersection with the geodesic η is compact (see Fig. 4.) Moreover, the distance of points $\eta(s)$ on this geodesic from this lunette region tends to infinity as $s \rightarrow \infty$. From this it is clear that the balls $B(Q_j, R_1)$ are eventually disjoint from Ω . By the choice of height h , the truncated cylinders $B(Q_j, R_1) \times [-C_2, C_2]$ are then disjoint from $R(\hat{\sigma}, -h', h')$ as claimed.

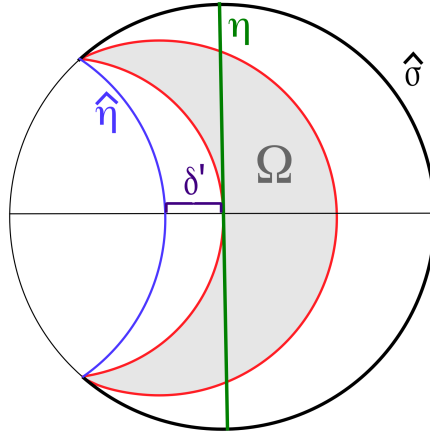


FIGURE 4. The lunette region Ω between the core curve $R(\hat{\sigma}, -h', h') \cap \{t = 0\}$ and the equidistant curve which is the projection to \mathbb{H}^2 of $R(\hat{\sigma}, -h', h') \cap \{t = \pm 2C_2\}$.

Now replace the vertical planes P_j with the tall rectangles $R_j := T_j^{-1}(R(\hat{\sigma}, -h', h'))$. These meet E vertically at (q_j, t_j) and are disjoint from $(B(o, R_1) \times \mathbb{R}) \cap E$, for j large. The rest of the proof then proceeds exactly as before. It is possible to apply the maximum principle because there are (many) foliations of $\mathbb{H}^2 \times \mathbb{R}$ by families of tall rectangles of which R_j is one element.

This exhausts all eventualities. The proof is now complete. \square

We conclude this section with a closely related result about the shape of the set of points on A where the tangent plane is vertical.

Proposition 5.4. *Let A be a properly Alexandrov-embedded, minimal annulus such that $\Pi(A) = (\gamma^+, \gamma^-)$ consists of two $\mathcal{C}^{2,\alpha}$ graphs over \mathbb{S}^1 . Then the set V of all points on A where the tangent plane is vertical (or equivalently, where the normal has no vertical component) is a regular curve which generates $H_1(A, \mathbb{Z})$. Moreover,*

the Gauss map of A restricted to V , $\nu|_V$, is a diffeomorphism from V to the equator of the unit sphere \mathbb{S}^2 .

Proof. Suppose H_s is a smooth ‘sweepout’ of $\mathbb{H}^2 \times \mathbb{R}$ by vertical planes. In other words, the H_s are leaves of a smooth foliation. Assuming that the parameter s varies over \mathbb{R} , we define a height function $K : \mathbb{H}^2 \times \mathbb{R} \rightarrow \mathbb{R}$ by setting $H_s = \{K = s\}$. Let K_A denote the restriction of K to A . We claim that K_A is a Morse function with precisely two critical points, each of index 1.

The graphical representation theorem proved in this section shows that for s very negative, $H_s \cap A$ is a union of two arcs, each one lying in an end of A . In fact, for any value of s , $H_s \cap A$ intersects a neighborhood of $\partial\mathbb{H}^2 \times \mathbb{R}$ in four simple arcs, two arriving in γ^+ and two arriving in γ^- . Now as s increases from $-\infty$ there is a point of first tangency with A , say at $s = s_1$, which occurs at a point $p_1 \in V$. As in the previous proof, the intersection $H_{s_1} \cap A$ is then a network of curves on H_{s_1} , which at the point of intersection is a union of ℓ curves intersecting at equal angles for some $\ell \geq 2$.

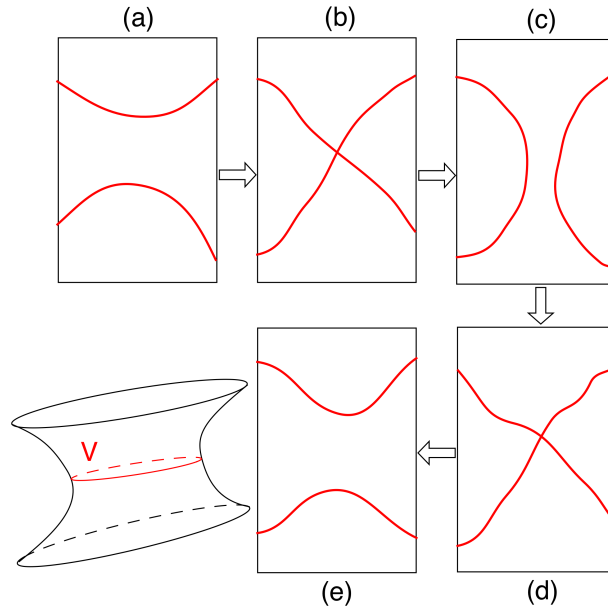


FIGURE 5. Evolution of $H_s \cap A$ when there are no loops.

If there is not any closed loop then the shape of $H_{s_1} \cap A$ is as in Figure 5-(b). In particular $\ell = 2$. If there is σ a closed loop in H_{s_1} , then (by the maximum principle) σ cannot bound a compact disk in A . So σ generates $H_1(A, \mathbb{Z})$. By the maximum

principle again, there is only one such loops. In particular, $\ell \leq 3$. If $\ell = 3$ then σ separates γ^+ and γ^- and so it separates the diverging arcs in $H_{s_1} \cap A$, two on each region of $H_{s_1} \setminus \sigma$. But this is absurd because one of the components of $H_{s_1} \setminus \sigma$ is compact. Hence $\ell = 2$ and the shape of $H_{s_1} \cap A$ is either as in Figure 6-(b) or as in Figure 7-(c).

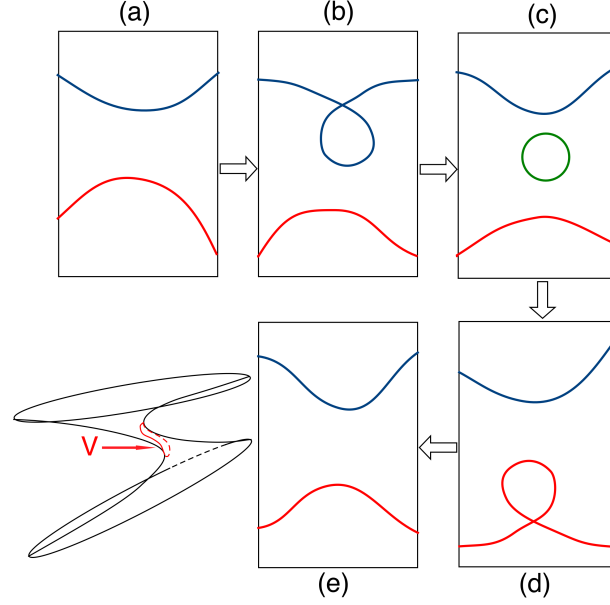


FIGURE 6. Possible evolution of $H_s \cap A$ and the curve σ , when σ is a generator of $H_1(A, \mathbb{Z})$.

In any case $\ell = 2$, and hence this is a simple tangency, or in other words, the function K_A has a nondegenerate critical point of index 1 at s_1 . Letting s increase further, we encounter some number of other critical points p_2, \dots, p_r , at the values $s = s_2, \dots, s_r$, each one of which corresponds to another nondegenerate critical point of index 1 of K_A .

To conclude, observe that we can apply the standard Morse-theoretic arguments to see how these critical points correspond to a decomposition of A into a union of cells.

In the first case (Figure 5), for s very negative, $A \cap \{K_A \leq s\}$ is a union of two disks. The transition between the sublevel $K_A \leq s_1 - \epsilon$ and $K_A \leq s_1 + \epsilon$ corresponds to attaching a two-cell which connects these two disks, resulting in another (topological) disk. Crossing the next critical point, another two-cell is added, which

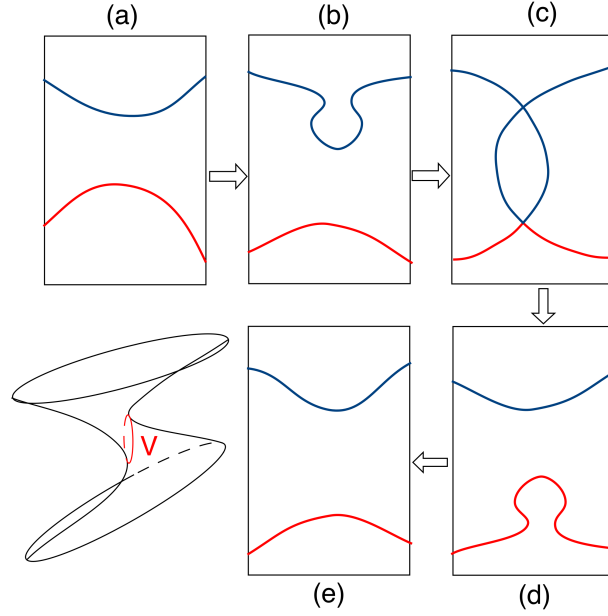


FIGURE 7. The other possible evolution of $H_s \cap A$ and the curve σ , when it contains a generator of $H_1(A, \mathbb{Z})$.

changes the topology again. Each of the remaining critical points add further handles. However, since A is an annulus, and in particular has genus 0, we must have $r = 2$. Hence there are precisely two critical points.

In the second case (Figure 6), again we have that for s very negative, $A \cap \{K_A \leq s\}$ is a union of two disks. Now, the transition between the sublevel $K_A \leq s_1 - \epsilon$ and $K_A \leq s_1 + \epsilon$ corresponds to attaching a two-cell which connects one of the disks with itself, resulting in a topological annulus. Crossing the next critical point, another two-cell is added, which connects the annulus with the other disk. Again we deduce that $r = 2$. Using similar arguments we deduce that $r = 2$ in the last case (Figure 7).

Note that since both these critical points are nondegenerate, the set of points $p \in V$ near either p_1 or p_2 constitute a regular curve.

We may of course do this for any sweepout of $\mathbb{H}^2 \times \mathbb{R}$ by vertical planes. This shows that in any direction, there are precisely two critical points and hence V is a regular curve which generates $H_1(A, \mathbb{Z})$. Since each H_s where tangency occurs is a support plane for \mathcal{O} , we conclude also that the projected domain \mathcal{O} is convex. \square

6. THE MANIFOLD OF MINIMAL ANNULI

In this section we prove the basic structural result about the space of minimal annuli.

Theorem 6.1. *The space \mathfrak{A}' of properly Alexandrov-embedded minimal annuli with $\mathcal{C}^{2,\alpha}$ boundary curves is a Banach submanifold in the space of complete properly embedded surfaces of class $\mathcal{C}^{2,\alpha}$ in $\mathbb{H}^2 \times \mathbb{R}$.*

Proof. Fix any $A \in \mathfrak{A}'$ and let u be a sufficiently small $\mathcal{C}^{2,\alpha}$ function on A . We denote by A_u the normal graph over A by u , i.e.,

$$A_u = \{\exp_p(u(p)\nu(p)) : p \in A\},$$

where ν is the unit normal field to A . The second order nonlinear elliptic differential operator

$$N(u) = u_{xx}(1 + x^2 u_y^2) - 2x^2 u_x u_y u_{xy} + u_{yy}(1 + x^2 u_x^2) - x u_x (u_x^2 + u_y^2)$$

calculates the mean curvature of A_u , (we use upper half-space coordinates (x, y) on \mathbb{H}^2 with $x > 0$ rather than polar coordinates since the formula is simpler) and minimal normal graphs correspond to solutions u to $N(u) = 0$. In fact, N is nondegenerate because by a fortuitous accident the ‘geometric’ mean curvature operator \mathcal{N} contains an overall factor of $\frac{1}{4}(1 - r^2)^2$, which we can remove.

The linearization of \mathcal{N} equals the ‘geometric’ Jacobi operator

$$\mathcal{L}_A = \Delta_A + |S|^2 + \text{Ric}(\nu, \nu),$$

where S is the shape operator (or second fundamental form) of A . This equals the same degenerate factor $\frac{1}{4}(1 - r^2)^2$ times the linearization $DN|_{u=0} := L$, i.e., $\mathcal{L} = \frac{1}{4}(1 - r^2)^2 L$.

The net effect is that we can invoke standard consequences of ellipticity rather than the uniformly degenerate elliptic theory from [8], which is necessary for the detailed study of operators with the type of degeneracy as \mathcal{L} . Note also that \mathcal{L} is self-adjoint with respect to the area form $d\sigma_A$, while L is symmetric with respect to the area form $d\sigma_D$ for the standard Euclidean metric on the disk, and self-adjoint with respect to Dirichlet boundary conditions.

The main results about N and L are as follows.

Proposition 6.2. *Let $A \in \mathfrak{A}'$ and suppose that $\partial A = \Gamma$ consists of a pair γ^\pm of $\mathcal{C}^{2,\alpha}$ horizontal curves. Then*

- i) *The graph function $u \in \mathcal{C}^{2,\alpha}(\overline{A})$;*
- ii) *If ϕ is a solution to $L\phi = 0$ with boundary values $\phi_0^\pm \in \mathcal{C}^{2,\alpha}(\mathbb{S}^1)$, then $\phi \in \mathcal{C}^{2,\alpha}(\overline{A})$;*

- iii) The operator $L : \mathcal{C}_D^{2,\alpha}(A) \longrightarrow \mathcal{C}^{0,\alpha}(A)$ is Fredholm of index 0, where $\mathcal{C}_D^{2,\alpha}$ is the space of $\mathcal{C}^{2,\alpha}$ functions which vanish at ∂A . Its kernel equals $\mathfrak{J}^0(A)$, and this same finite dimensional space is a complement for its range;
- iv) If $\phi \in \mathfrak{J}^0$, i.e., $\phi_0^\pm = 0$, then $\phi \in \mathcal{C}^\infty(\overline{A})$.

We say that A is nondegenerate if $\mathfrak{J}^0(A) = \{0\}$.

Proposition 6.3. *If $A \in \mathfrak{A}'$ is nondegenerate, then there exists a neighborhood \mathcal{U} of 0 in $\mathcal{C}^{2,\alpha}(\mathbb{S}^1)^2$ and a smooth map $G : \mathcal{U} \rightarrow \mathcal{C}_D^{2,\alpha}(A)$ such that $N((\phi_0^+, \phi_0^-) + G((\phi_0^+, \phi_0^-))) = 0$, and all solutions u to $N(u) = 0$ sufficiently close to 0 are of this form.*

The proof is a standard application of the implicit function theorem to the map

$$\mathcal{C}^{2,\alpha}(\mathbb{S}^1)^2 \times \mathcal{C}_D^{2,\alpha}(A) \ni ((\phi_0^+, \phi_0^-), w) \longrightarrow N(e(\phi_0^+, \phi_0^-) + w) \in \mathcal{C}^{0,\alpha}(A).$$

Here $(\phi_0^+, \phi_0^-) \mapsto e(\phi_0^+, \phi_0^-)$ is a continuous extension operator $\mathcal{C}^{2,\alpha}(\mathbb{S}^1)^2 \rightarrow \mathcal{C}^{2,\alpha}(A)$.

To prove that \mathfrak{A}' is a Banach manifold even around degenerate annuli, it is necessary to characterize those pairs (ϕ_0^+, ϕ_0^-) which occur as leading coefficients of elements of $\mathfrak{J}(A)$.

Proposition 6.4. *Let (ϕ_0^+, ϕ_0^-) be a pair of functions in $(\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2$. Then, there exists a Jacobi field $\phi \in \mathfrak{J}(A)$ satisfying $\phi|_{\partial A^\pm} = \phi_0^\pm$ if, and only if,*

$$\int_{\mathbb{S}^1} (\phi_0^+ \psi_1^+ + \phi_0^- \psi_1^-) = 0$$

for every $\psi \in \mathfrak{J}^0(A)$.

Proof. If $\phi \in \mathfrak{J}(A)$ and $\psi \in \mathfrak{J}^0(A)$, then

$$\begin{aligned} 0 &= \int_A (L\phi)\psi - \phi(L\psi) = \int_{\mathbb{S}^1} (\phi_0^+ \psi_1^+ - \phi_1^+ \psi_0^+) + (\phi_0^- \psi_1^- - \phi_1^- \psi_0^-) \\ &= \int_{\mathbb{S}^1} (\phi_0^+ \psi_1^+ + \phi_0^- \psi_1^-). \end{aligned}$$

(The integrals at the two boundary components appear with the same sign because we are using the outward pointing normal derivative at each of these.)

This necessary condition is also sufficient. Indeed, fix any ϕ_0^\pm satisfying this orthogonality condition, and set $u = e(\phi_0^+, \phi_0^-)$. Then $Lu = f \in \mathcal{C}^{0,\alpha}$. By part iii) of Proposition 6.2, there exists $\psi \in \mathfrak{J}^0(A)$ and $v \in \mathcal{C}_D^{2,\alpha}(A)$ such that $Lv = f + \psi$. Thus writing $\phi = v - u$, then $L\phi = \psi$. We now show that this is impossible unless $\psi = 0$. Indeed, since v vanishes at ∂A , $u^\pm = \phi_0^\pm$. Now we compute that

$$(6.1) \quad \int_A |\psi|^2 = \int_A (L\phi)\psi - \phi(L\psi) = \int_{\partial_+ A} \phi_0^+ \psi_1^+ + \int_{\partial_- A} \phi_0^- \psi_1^- = 0,$$

hence $\psi = 0$. This completes the proof. \square

This result proves that the set of pairs (ϕ_0^+, ϕ_0^-) which can occur as leading coefficients of Jacobi fields $\phi \in \mathfrak{J}(A)$ has finite codimension in $\mathcal{C}^{2,\alpha}(\mathbb{S}^1)^2$, and that a good choice of complementary subspace for it is the space

$$W = \{e(\phi_1^+, \phi_1^-) : \phi \in \mathfrak{J}^0(A)\}$$

of normal derivatives of all elements of $\mathfrak{J}^0(A)$.

Proposition 6.5. *The map*

$$(6.2) \quad L : W \oplus \mathcal{C}_D^{2,\alpha}(A) \longrightarrow \mathcal{C}^{0,\alpha}(A)$$

is surjective, with nullspace $\mathfrak{J}^0(A)$.

Proof. We have already noted that the range of L on $\mathcal{C}_D^{2,\alpha}$ is a finite codimensional space in $\mathcal{C}^{0,\alpha}$ complementary to $\mathfrak{J}^0(A)$. Suppose then that $\gamma \in \mathfrak{J}^0(A)$ and

$$\int_A \gamma(L(e(\phi_1^+, \phi_1^-) + u)) = 0 \quad \text{for every } \phi \in \mathfrak{J}^0(A) \text{ and } u \in \mathcal{C}_D^{2,\alpha}(A).$$

Taking $\phi = 0$ and integrating by parts simply confirms that $L\gamma = 0$. Next, using that γ vanishes at the boundary, let $u = 0$ and integrate by parts again to obtain

$$0 = \int_A \gamma(Le(\phi_1^+, \phi_1^-)) - (L\gamma)e(\phi_1^+, \phi_1^-) = \int_{\mathbb{S}^1} (\gamma_1^+ \phi_1^+ + \gamma_1^- \phi_1^-).$$

Letting $\phi = \gamma$ shows that $\gamma_1^\pm = 0$, and hence that $\gamma = 0$.

To finish the proof, note that if $\psi = e(\phi_1^+, \phi_1^-) + u \in \mathfrak{J}(A)$, then $\psi_0^\pm = \phi_1^\pm$ for some $\phi \in \mathfrak{J}^0(A)$, which we showed above is impossible unless $\phi = 0$. This proves that the null space of (6.2) equals $\mathfrak{J}^0(A)$. \square

We may now complete the proof of Theorem 6.1. The case when A is nondegenerate has already been handled, so suppose that $\mathfrak{J}^0(A) \neq \{0\}$. Choose subspaces $\mathfrak{J}^0(A)^\perp \subset \mathfrak{J}(A)$ and $\mathcal{X}_0 \subset \mathcal{C}_D^{2,\alpha}(A)$, each complementary to $\mathfrak{J}^0(A)$ in the respective larger ambient spaces. Immediately from Proposition 6.5,

$$L : \mathfrak{J}^0(A)^\perp \oplus W \oplus \mathcal{X}_0 \longrightarrow \mathcal{C}^{0,\alpha}(A)$$

is surjective, with nullspace $\mathfrak{J}^0(A)^\perp$. In addition,

$$N : \mathfrak{J}^0(A)^\perp \oplus W \oplus \mathcal{X}_0 \longrightarrow \mathcal{C}^{0,\alpha}(A)$$

is well-defined and smooth. The implicit function theorem implies, as before, the existence of a map

$$G : \mathfrak{J}^0(A)^\perp \longrightarrow W \oplus \mathcal{X}_0$$

and a neighbourhood \mathcal{W} of 0 in $\mathfrak{J}^0(A)^\perp$ such that

$$\mathcal{W} \ni \phi \mapsto N(\phi + G(\phi)) \equiv 0,$$

and all solutions of N near to 0 are of this form.

Once again, this is a chart for \mathfrak{A}' near A , which proves that \mathfrak{A}' is a Banach submanifold even around degenerate points. \square

Remark 6.6. Since it will be important later, we record that there are explicit expressions for the decaying Jacobi fields $\mathfrak{J}^0(A)$ associated to the rotationally invariant catenoid $A_0 = C_h$. We calculate from these that the space of normal derivatives of elements of $\mathfrak{J}^0(A_0)$ is spanned by $(\sin \theta, \sin \theta)$ and $(\cos \theta, \cos \theta)$.

7. THE EXTENDED BOUNDARY PARAMETRIZATION

Let A be a proper, Alexandrov-embedded, minimal annulus such that $\Pi(A) = (\gamma^+, \gamma^-)$ consists of two $\mathcal{C}^{2,\alpha}$ graphs over \mathbb{S}^1 . The bottom boundary curve γ^- bounds a unique minimal disk D^- ; this is the vertical graph of a function v^- . Let u^- denote the function parametrizing the bottom end of A . We shall consider the space of minimal annuli $\mathfrak{A}^* \subset \mathfrak{A}'$ which satisfy

$$(7.1) \quad u_r^-(1, \theta) - v_r^-(1, \theta) < 0, \quad \forall \theta \in \mathbb{S}^1.$$

Clearly \mathfrak{A}^* is an open subset of \mathfrak{A}' , and hence its tangent space $T_A \mathfrak{A}^*$ at any point equals $\mathfrak{J}(A)$. In addition, it is trivial that $\mathfrak{A} \subset \mathfrak{A}^*$. Consider the map

$$\Pi : \mathfrak{A}^* \rightarrow \mathcal{C}^{2,\alpha}(\mathbb{S}^1)^2,$$

which takes any $A \in \mathfrak{A}^*$ to its pair of boundary curves $(\gamma^+, \gamma^-) \in \mathcal{C}^{2,\alpha}(\mathbb{S}^1)^2$.

The perhaps naive hope is that this map can be used to parametrize \mathfrak{A}^* by some subset of $\mathcal{C}^{2,\alpha}(\mathbb{S}^1)^2$. To understand whether this is feasible, the first step is to compute its index.

Theorem 7.1. *The map $\Pi : \mathfrak{A}^* \rightarrow \mathcal{C}^{2,\alpha}(\mathbb{S}^1) \times \mathcal{C}^{2,\alpha}(\mathbb{S}^1)$ is Fredholm of index zero.*

Proof. The assertion is that the linear map $D\Pi|_A$ is Fredholm of index 0 for every A . However, $D\Pi|_A(\phi) = \phi_0$, the leading coefficient of the Jacobi field ϕ at ∂A , so we must show that $\mathfrak{J}(A) \ni \phi \mapsto \phi_0 \in \mathcal{C}^{2,\alpha}(\mathbb{S}^1)^2$ is Fredholm of index 0. This follows immediately from Proposition 6.2. \square

We have already seen that $D\Pi$ is not invertible at the catenoid C_h . It has a two-dimensional nullspace there, and the implicit function theorem shows that the range of Π is contained locally near $\Pi(C_h)$ around a codimension 2 submanifold. We prove later that this image has nontrivial interior, but by Proposition 3.3, $\Pi(C_h)$ is not an interior point of this image. In fact, we do not have a precise characterization of $\Pi(\mathfrak{A}^*)$.

It is useful to define a slightly different boundary correspondence via the extended boundary map

$$(7.2) \quad \begin{aligned} \tilde{\Pi} : \mathfrak{A}^* \times \mathbb{R} \times \mathbb{C} &\longrightarrow \mathcal{C}^{2,\alpha}(\mathbb{S}^1)^2 \times \mathbb{R} \times \mathbb{C}, \\ \tilde{\Pi}(A, a, \eta) &= (\Pi_-(A), \Pi_+(A) + a + \operatorname{Re}(\eta e^{i\theta}), G(A)). \end{aligned}$$

Here $\Pi_{\pm}(A) = \gamma_{\pm}$ and the components (G_0, G_1, G_2) of G are defined as follows. The bottom boundary curve γ_- bounds a unique minimal disk D^- ; this is the vertical graph of a function v^- . Letting u^- denote the function parametrizing the bottom end of A , we write

$$\begin{aligned} f_0(A) &= \operatorname{Flux}(A, \partial_- A, E_3) = \int_{\mathbb{S}^1} u_r^-(1, \theta) d\theta, \\ f_1(A) + i f_2(A) &= \int_{\mathbb{S}^1} e^{i\theta} (u_r^-(1, \theta) - v_r^-(1, \theta)) d\theta, \end{aligned}$$

and in terms of these, define

$$G_0(A) = f_0(A) - f_0(A)^{-1}, \quad G_1(A) + i G_2(A) = (f_1(A) + i f_2(A)) / f_0(A).$$

Definition 7.2. We refer to $\mathbf{C}(A) := G_1(A) + i G_2(A)$ as the **center** of $A \in \mathfrak{A}^*$.

Note that

$$(7.3) \quad \mathbf{C}(\tilde{R}_{\zeta}(A)) = R_{\zeta}(\mathbf{C}(A)),$$

where $R_{\zeta} : \mathbb{R} \times \mathbb{C} \rightarrow \mathbb{R} \times \mathbb{C}$ is the rotation $(t, z) \mapsto (t, e^{i\zeta} z)$.

Since the flux of the disk D^- is zero, we can also write

$$f_0(A) = \int_{\mathbb{S}^1} (u_r^-(1, \theta) - v_r^-(1, \theta)) d\theta,$$

and certainly $u_r^-(1, \theta) - v_r^-(1, \theta) < 0$ for all $\theta \in \mathbb{S}^1$. We thus have that $|\mathbf{C}(A)| < 1$ for all $A \in \mathfrak{A}^*$, i.e., $\mathbf{C}(A) \in \mathbb{H}^2$. Now define

$$(7.4) \quad \widetilde{\mathfrak{A}^*} := \tilde{\Pi}^{-1}(\mathcal{C}^{2,\alpha}(\mathbb{S}^1)^2 \times \mathbb{R} \times \mathbb{D}).$$

Remark 7.3. Evaluating $f_1 + i f_2$ on the 3-dimensional family of catenoids \mathcal{M} (see Remark 2.1), then $v_r^- \equiv 0$, and hence

$$f_1(C_{h,z_0}) + i f_2(C_{h,z_0}) = \int_{\mathbb{S}^1} e^{i\theta} u_r^-(1, \theta) d\theta.$$

It is then straightforward to check that $z_0 = G_1(C_{h,z_0}) + i G_2(C_{h,z_0})$.

This remark shows that the center of the neck of the catenoid C_{h,z_0} in the obvious geometric sense equals $\mathbf{C}(C_{h,z_0})$. We show in Section 8 that $\mathbf{C}(A)$ behaves like this center more generally in the following sense. If A_n is a sequence of annuli for which the sequence of vertical fluxes is bounded and $\mathbf{C}(A_n)$ diverges in \mathbb{H}^2 , then A_n

converges to two disjoint minimal disks, and the necks of these annuli disappear at infinity.

The motivation for introducing this enhanced boundary map $\tilde{\Pi}$ is that the catenoids are no longer degenerate points.

Theorem 7.4. *The extended boundary correspondence $\tilde{\Pi}$ is a proper Fredholm map of index 0. It is locally invertible near any one of the catenoids C_h .*

Theorem 7.4 will be proved in a series of steps. In the remainder of this section we verify that $\tilde{\Pi}$ is Fredholm of index 0 and check that its differential at C_h is invertible. The properness assertion is more difficult and its proof occupies the next section.

The main result of this paper is an essentially global existence theorem which is proved using degree theory. This relies on the fact that a proper Fredholm map between Banach manifolds has a \mathbb{Z} -valued degree. The importance of Theorem 7.4 is that it implies that this degree equals 1. This will be discussed carefully below.

Proposition 7.5. *The map $\tilde{\Pi}$ is Fredholm of index 0.*

Proof. This map is Fredholm because its domain and range are finite dimensional extensions of those for Π ; since these extensions have the same dimension, the index remains 0. \square

Proposition 7.6. *Let C be any catenoid. Then $D\tilde{\Pi}|_C$ is invertible.*

Proof. We may as well suppose that $C = C_h$ is centered. Since the index of this differential vanishes, it suffices to show that its nullspace is trivial. Suppose then that $D\tilde{\Pi}|_C(\phi, \alpha, \mu) = (0, 0)$ for some $(\phi, \alpha, \mu) \in \mathfrak{J}(A) \times \mathbb{R} \times \mathbb{C}$. This corresponds to the set of equations

$$\phi_0^- = 0, \quad \phi_0^+ + \alpha + \operatorname{Re}(\mu e^{i\theta}) = 0, \quad \text{and} \quad DG_C(\phi) = 0.$$

The first equation states that the Jacobi field ϕ vanishes at $\partial_- A$, while the second condition shows that its restriction to $\partial_+ A$ lies in the span of $\{1, \cos \theta, \sin \theta\}$. On the other hand, by our knowledge of the elements of $\mathfrak{J}^0(C_h)$, $(\phi_0^+, \phi_0^-) = (a_0^+, a_0^-) + a_1(\cos \theta, \cos \theta) + a_2(\sin \theta, \sin \theta)$. Comparing these two expressions gives $a_1 = a_2 = 0$, and hence ϕ is the Jacobi field corresponding to the variation $\epsilon \mapsto C_{h+\epsilon}$ (where we assume that the bottom boundary of $C_{h+\epsilon}$ remains fixed while the height of the top circle varies). Combining this with the two-dimensional nullspace $\mathfrak{J}^0(C)$ of the map $\phi \mapsto \phi_0$, we see that the nullspace of the differential of the first two components of $D\tilde{\Pi}|_C$ is three-dimensional.

Now examine the equation $DG|_C(\phi) = 0$. First consider the Jacobi field ϕ with constant boundary values $\phi_0^- = 0$, $\phi_0^+ = a_0^+$. We compute that

$$DG_0|_C(\phi) = (1 + f_0(C)^{-2})Df_0|_C(\phi) = c \int_{\mathbb{S}^1} \phi_1^+(\theta) d\theta,$$

with $c \neq 0$. We claim that this expression is nonzero. Indeed, $\phi_1^+(\theta) = \partial_r \phi(1, \theta)$ at $\partial_+ A$, and by the maximum principle, this normal derivative is nonnegative. Thus this whole expression vanishes if and only if $\partial_r \phi \equiv 0$ at this top boundary. Hence $DG_0|_C(\phi) = 0$ implies $a_0 = 0$.

Next compute that

$$DG_j|_C = \frac{f_0(C)Df_j|_C - f_j(C)Df_0|_C}{f_0(C)^2}, \quad j = 1, 2.$$

Since $f_j(C) = 0$, it suffices to check that the two-by-two matrix which is the restriction of the Jacobian of (f_1, f_2) to $\mathfrak{J}^0(C)$ is nonzero. However, this is clear from Remark 7.3 and the formulæ

$$D(f_1 + i f_2)|_C(\phi) = \int_{\mathbb{S}^1} e^{i\theta} \phi_1^-(\theta) d\theta,$$

since $\phi_1^- = c_1 \cos \theta + c_2 \sin \theta$ if $\phi \in \mathfrak{J}^0(C)$. This proves that $D\tilde{\Pi}|_C$ is invertible. \square

Proposition 7.7. *For any $h \in (0, \pi)$, $\tilde{\Pi}^{-1}(\tilde{\Pi}(C_h, 0, 0, 0)) = \{(C_h, 0, 0, 0)\}$.*

Proof. After a vertical translation, we may assume that $\tilde{\Pi}(C_h, 0, 0, 0) = (\mathbb{S}^1 \times \{0, h\}, G_0(C_h), 0, 0)$. If $(A, x_0, x_1, x_2) \in \tilde{\Pi}^{-1}(\mathbb{S}^1 \times \{0, h\}, G_0(C_h), 0, 0)$, then by definition, $\partial^- A = \mathbb{S}^1 \times \{0\}$ and $\partial^+ A = \{(\theta, h + \alpha_0 + \alpha_1 \cos \theta + \alpha_2 \sin \theta) : \theta \in \mathbb{S}^1\}$, for some $\alpha \in \mathbb{R}^3$. Using Proposition 3.3 and facts that $h \mapsto G_0(C_h)$ is bijective and $(G_1(A), G_2(A)) = (0, 0)$, we deduce that $\alpha = (0, 0, 0)$ and $A = C_h$. \square

8. COMPACTNESS

Our goal in this section is to prove that the map $\tilde{\Pi} : \widetilde{\mathfrak{A}}^* \rightarrow \mathcal{C}^{2,\alpha}(\mathbb{S}^1)^2 \times \mathbb{R} \times \mathbb{D}$ is proper. In other words, we show that if $\tilde{\Pi}(A_n, x^{(n)}) = (\gamma_n^-, \gamma_n^+, z^{(n)})$ converges in $\mathcal{C}^{2,\alpha}(\mathbb{S}^1)^2 \times \mathbb{R} \times \mathbb{D}$, then some subsequence of $(A_n, x^{(n)})$ converges in $\widetilde{\mathfrak{A}}^*$. The first subsection below addresses the properness of $\tilde{\Pi}$, while the second considers the various modes of divergence of sequences in the space of properly embedded minimal annuli \mathfrak{A} . The former study of these modes of divergence will allow us to obtain properness of the natural projection Π , when we restrict it to certain submanifolds and open regions of \mathfrak{A} .

8.1. The properness of the map $\tilde{\Pi}$. We conclude this section with the proof of the properness of $\tilde{\Pi} : \widetilde{\mathfrak{A}}^* \rightarrow \mathcal{C}^{2,\alpha}(\mathbb{S}^1)^2 \times \mathbb{R} \times \mathbb{D}$, where, as introduced earlier,

$$\widetilde{\mathfrak{A}}^* = \tilde{\Pi}^{-1}(\mathcal{C}^{2,\alpha}(\mathbb{S}^1)^2 \times \mathbb{R} \times \mathbb{D}).$$

Lemma 8.1. *Let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence in \mathfrak{A}^* such that:*

- $\{f_0(A_n)\} \rightarrow h_0 < 0$,

- $\{\partial_- A_n\} \rightarrow \gamma_0^-$ in $\mathcal{C}^{2,\alpha}(\mathbb{S}^1)$,
- the bottom ends of $\{A_n\}$ smoothly converge to D_0^- on $\overline{\mathbb{H}^2 \times \mathbb{R}} \setminus E$, where E is a vertical line in $\partial(\mathbb{H}^2 \times \mathbb{R})$.

Then the sequence of centers $\mathbf{C}(A_n)$ diverges in \mathbb{H}^2 , i.e., $|\mathbf{C}(A_n)| \rightarrow 1$.

Proof. Let $e^{i\theta_0}$ denote the point of intersection $\mathbb{S}^1 \cap E$. Rotating, we can assume that $0 < \theta_0 < \pi/2$.

From our hypotheses, we know that $|(u_n^-)_r - (v_n^-)_r| < 1/n$, in an arc L_n of $\mathbb{S}^1 \setminus \{e^{i\theta_0}\}$, with $\cup_n L_n = \mathbb{S}^1 \setminus \{e^{i\theta_0}\}$. Label $L'_n = \mathbb{S}^1 - L_n$. If n is large enough, then there are angles $0 < \theta_1^n < \theta_0 < \theta_2^n < \pi/2$, such that $L'_n = \{e^{i\theta} : \theta_1^n < \theta < \theta_2^n\}$. Hence

$$G_1(A_n) = \frac{\int_{\mathbb{S}^1} \cos \theta ((u_n^-)_r - (v_n^-)_r) d\theta}{\int_{\mathbb{S}^1} ((u_n^-)_r - (v_n^-)_r) d\theta} = \frac{\int_{L'_n} \cos \theta ((u_n^-)_r - (v_n^-)_r) d\theta + \int_{L_n} \cos \theta ((u_n^-)_r - (v_n^-)_r) d\theta}{\int_{L'_n} ((u_n^-)_r - (v_n^-)_r) d\theta + \int_{L_n} ((u_n^-)_r - (v_n^-)_r) d\theta}$$

and so

$$\begin{aligned} & \frac{\cos(\theta_1^n) \int_{L'_n} ((u_n^-)_r - (v_n^-)_r) d\theta + \int_{L_n} \cos \theta ((u_n^-)_r - (v_n^-)_r) d\theta}{\int_{L'_n} ((u_n^-)_r - (v_n^-)_r) d\theta + \int_{L_n} ((u_n^-)_r - (v_n^-)_r) d\theta} \geq G_1(A_n) \geq \\ & \frac{\cos(\theta_2^n) \int_{L'_n} ((u_n^-)_r - (v_n^-)_r) d\theta + \int_{L_n} \cos \theta ((u_n^-)_r - (v_n^-)_r) d\theta}{\int_{L'_n} ((u_n^-)_r - (v_n^-)_r) d\theta + \int_{L_n} ((u_n^-)_r - (v_n^-)_r) d\theta}. \end{aligned}$$

Since the integrals over L_n converge to 0 and

$$\left\{ \int_{L'_n} ((u_n^-)_r - (v_n^-)_r) d\theta \right\} \rightarrow h_0,$$

we deduce that $\{G_1(A_n)\} \rightarrow \cos \theta_0$. Similarly, $\{G_2(A_n)\} \rightarrow \sin \theta_0$. \square

Lemma 8.2. *Let $A_n \in \mathfrak{A}^*$ be a sequence such that $\Pi(A_n)$ converges to a pair of curves $\Gamma_0 = (\gamma_0^+, \gamma_0^-) \in \mathcal{C}^{2,\alpha}(\mathbb{S}^1)^2$, $\{G_0(A_n)\}$ is bounded in \mathbb{R} , and finally that the curves $V_n = \{p \in A_n : \langle \nu_n(p), E_3 \rangle = 0\}$ remain in a compact region K of $\mathbb{H}^2 \times \mathbb{R}$. Then, up to a subsequence, A_n converges to a minimal annulus $A_0 \in \mathfrak{A}^*$ with $\Pi(A_0) = \Gamma_0$. The convergence is smooth on the interior and $\mathcal{C}^{2,\alpha}$ up to the boundary.*

Proof. We know that each $A_n \setminus (A_n \cap K)$ is a union of two graphs. Using classical elliptic estimates and the Arzelà-Ascoli theorem, these two sequences of graphs converge smoothly to minimal graphs for which the boundaries at infinity are γ_0^\pm .

Note that $\gamma_0^+ \neq \gamma_0^-$. Indeed, if this were the case, the sequence of minimal annuli A_n would converge to the minimal disk D_0 spanned by $\gamma_0^+ = \gamma_0^-$. This would force

the vertical flux $f_0(A_n)$ to converge to 0, and hence $G_0(A_n) \rightarrow \infty$, contrary to assumption.

Next, we claim that $A_n \cap K$ must converge smoothly to a regular annulus with boundary inside K . To prove this, we use again that the vertical flux $f_0(A_n)$ is bounded away from 0. The key point is that it is impossible for $A_n \cap K$ to ‘pinch’. Suppose that this were to occur. Then there would exist points $p_n \in K \cap A_n$ at which the shape operator S_n of A_n satisfies

$$\lambda_n := |S_n(p_n)| = \max\{|S_n|(p) : p \in K \cap A_n\} \longrightarrow +\infty.$$

Then the rescaled surfaces $\frac{1}{\lambda_n}(A_n - p_n)$ would converge to a complete minimal surface A_∞ in \mathbb{R}^3 which passes through the origin and with $|S_\infty(0)| = 1$. From Proposition 5.4 we have that the Gauss $\nu_\infty : A_\infty \rightarrow \mathbb{S}^2$ takes each value in the equator at most once. Moreover, A_∞ has the topology of either a disk or an annulus. Using a result by Mo and Osserman [13], A_∞ has finite total curvature -4π . Then A_∞ is either a catenoid or a copy of Enneper’s surface. However, Enneper’s surface is not Alexandrov-embedded, so it must be a catenoid. Thus at each point where $|S_n|$ blows up, a catenoidal neck is forming. This cannot happen at more than one point, since if this were to occur at two distinct points, there would be an enclosed annular region for which both boundary curves are very short, and this violates the isoperimetric inequality. Denoting this point of curvature blowup by p_0 , then away from p_0 , A_n converges to the union of two disks D_0^+ and D_0^- , each of which is a graph over \mathbb{H}^2 . Hence the sequence of curves in the statement of the theorem must converge to p_0 , and they must have bounded length. However, the length of V_n equals the vertical flux $\text{Flux}(A_n, V_n, E_3)$, and this convergence would force this flux to tend to 0, which we have assumed is not the case.

We have now proved that A_n must converge to an Alexandrov-embedded minimal annulus. It is clear from the convergence outside K that $\Pi(A_0) = \Gamma_0$. \square

Lemma 8.3. *Let $\{A_n\}$ be a sequence in $\widetilde{\mathfrak{A}}^*$ satisfying that:*

- $\Pi(A_n)$ converges to a pair of curves $\Gamma_0 = (\gamma_0^+, \gamma_0^-) \in \mathcal{C}^{2,\alpha}(\mathbb{S}^1)^2$.
- $\{G_0(A_n)\}$ is bounded in \mathbb{R} .
- The curves V_n have bounded length, but they diverge in $\mathbb{H}^2 \times \mathbb{R}$.

Then there exists a vertical line E contained in $\partial(\mathbb{H}^2 \times \mathbb{R})$ such that $\{A_n\}$ converges, smoothly on $\overline{\mathbb{H}^2 \times \mathbb{R}} \setminus E$, to $D_0^+ \cup D_0^-$, where D_0^+ and D_0^- are the minimal disks spanned by γ_0^+ and γ_0^- , respectively. In particular, the sequence of centers $\mathbf{C}(A_n)$ diverge in \mathbb{H}^2 .

Proof. Since the V_n diverge but their lengths remain bounded, their limit set is contained in a vertical line $E := \{q_\infty\} \times \mathbb{R}$ for some $q_\infty \in \mathbb{S}^1$. We know by Proposition 5.4 that $A_n \setminus V_n$ is the union of two graphs A_n^+ and A_n^- . Reasoning as in Theorem 8.11, the ends A_n^\pm converge as minimal graphs, smoothly in the interior and in $\mathcal{C}^{2,\alpha}$

on compact sets of $(\overline{\mathbb{H}^2} \setminus \{q_\infty\}) \times \mathbb{R}$, to the minimal disks D_0^\pm . A direct application of Lemma 8.1 gives that $|\mathbf{C}(A_n)| \rightarrow 1$. \square

Lemma 8.4. *Let $\{A_n\}$ be a sequence in \mathfrak{A}^* satisfying that:*

- $\Pi(A_n)$ converges to a pair of curves $\Gamma_0 = (\gamma_0^+, \gamma_0^-) \in \mathcal{C}^{2,\alpha}(\mathbb{S}^1)^2$.
- The length of curves $V_n = \{p \in A_n : \langle \nu_n(p), E_3 \rangle = 0\}$ diverges.

Then, $G_0(A_n) \rightarrow +\infty$.

Proof. The proof follows from the equality $\text{Flux}(A_n, V_n, E_3) = \int_{V_n} \langle E_3, \eta_{V_n} \rangle$ since the integrand is identically 1 and the lengths of the V_n tend to $+\infty$ (Theorem 8.17.) \square

We now prove the main result of this subsection.

Theorem 8.5. *Consider a sequence of elements $(A_n, x_n, y_n, z_n) \in \widetilde{\mathfrak{A}}^*$ such that*

$$(\widetilde{\Gamma}_n, \lambda_n, c_n) = \widetilde{\Pi}(A_n, x_n, y_n, z_n) \longrightarrow (\widetilde{\Gamma}_0, \lambda_0, c_0) \in \mathfrak{C} \times \mathbb{R} \times \mathbb{D}.$$

Then some subsequence of the (A_n, x_n, y_n, z_n) converges to $(A_0, x_0, y_0, z_0) \in \widetilde{\mathfrak{A}}$ and $\widetilde{\Pi}(A_0, x_0, y_0, z_0) = (\widetilde{\Gamma}_0, \lambda_0, c_0)$.

Proof. Since the sequence

$$\begin{aligned} (\lambda_n, c_n) &= (G_0(A_n), G_1(A_n) + i G_2(A_n)) = \\ &= \left(f_0(A_n) - f_0(A_n)^{-1}, \frac{f_1(A_n)}{f_0(A_n)} + i \frac{f_2(A_n)}{f_0(A_n)} \right) \end{aligned}$$

converges, we deduce that the sequence of vertical fluxes $\{f_0(A_n)\}_{n \in \mathbb{N}}$ converges to a positive constant μ_0 . As a consequence $\{(f_1(A_n), f_2(A_n))\}_{n \in \mathbb{N}}$ converges to a point $p_0 \in \mathbb{R}^2$. On the other hand, by assumption,

$$\widetilde{\Gamma}_n = (\gamma_n^-, \gamma_n^+ + x_n + y_n \cos \theta + z_n \sin \theta) \longrightarrow \widetilde{\Gamma}_0 = (\widetilde{\gamma}_0^-, \widetilde{\gamma}_0^+),$$

where $\gamma_n^\pm = \Pi_\pm(A_n)$. The first, easy, consequence is that $\gamma_n \rightarrow \gamma_0^- = \widetilde{\gamma}_0^-$.

Claim 8.6. *The sequence $\{\gamma_n^+\}$ also converges.*

To prove this, define $\omega_n := \sup \gamma_n^-$ and $\alpha_n := \inf \gamma_n^+$. First observe that if the parameters (x_n, y_n, z_n) are so large that $\omega_n := \alpha_n - \pi$, then we could bring some parabolic generalized catenoid to a point of tangency with A_n , which is impossible. However, this does not yet bound the individual components of this parameter set.

To do this, we show first that γ_n^+ cannot be too ‘tilted’, i.e., that $|(y_n, z_n)| \leq C$ for some C which depends on $\widetilde{\gamma}_0^+$.

If this is not the case, then γ_n^+ becomes increasingly tilted and converges as $n \rightarrow \infty$ to a vertical halfline which contains a point at distance less than π from γ_0^- . The limit of A_n then consists of the minimal disk D_0^- spanned by γ_0^- and a subset of a

vertical line E in $\partial(\mathbb{H}^2 \times \mathbb{R})$. We then apply Lemma 8.1 to deduce $\{\mathbf{C}(A_n)\}$ diverges in \mathbb{H}^2 , contrary to hypothesis.

This establishes that γ_n^+ also converges, and that $(x_n, y_n, z_n) \rightarrow (x_0, y_0, z_0)$, so $\gamma_n^+ \rightarrow \gamma_0^+ = \tilde{\gamma}_0^+ - x_0 - y_0 \cos \theta - z_0 \sin \theta$.

We need to prove finally that the sequence of annuli $\{A_n\}_{n \in \mathbb{N}} \subset \mathfrak{A}^*$ converges smoothly to an annulus $A_0 \in \mathfrak{A}^*$ and $\Pi(A_0) = (\gamma_0^-, \gamma_0^+)$. By the Lemmas 8.3 and 8.4, the curves $V_n = \{p \in A_n : \langle \nu_n(p), E_3 \rangle = 0\}$ remain in a compact region of $\mathbb{H}^2 \times \mathbb{R}$, because the vertical fluxes and the centers are bounded. So, we can apply Lemma 8.2 to deduce the existence of the $A_0 \in \mathfrak{A}^*$. \square

8.2. Diverging sequences in \mathfrak{A} . Suppose that $\{A_n\}$ is any sequence in \mathfrak{A} , whose sequence of boundary curves $\{\Pi(A_n)\}$ converges to $\Gamma = (\gamma^+, \gamma^-) \in \mathfrak{C}$. By Proposition 5.2, for each n there exists a solid cylinder $B(q_n, R_n) \times \mathbb{R}$ such that $A_n \setminus (B(q_n, R_n) \times \mathbb{R}) = E_n^\pm$ is the union of two vertical graphs $\mathbb{H}^2 \setminus B(q_n, R_n) \ni z \rightarrow (z, u_n^\pm(z))$, each one embedded. Up to a subsequence (which we assume without further comment), there are three possible behaviors:

Case I: Both the centers q_n and radii R_n can be chosen independent of n , hence E_n^\pm are graphs over a fixed annulus $\mathbb{H}^2 \setminus B(q, R)$;

Case II: The radii R_n are independent of n , but the centers q_n diverge;

Case III: The sequence of radii R_n diverges.

Our analysis relies on the following two results:

Theorem 8.7 (White [15]). *Let (Ω, g) be a Riemannian 3-manifold and $M_n \subset \Omega$ a sequence of properly embedded minimal surfaces with boundary such that*

$$\limsup_{n \rightarrow \infty} \text{length}\{\partial M_n \cap K\} < \infty,$$

for any relatively compact subset K of Ω . Define the area blowup set

$$Z := \{p \in \Omega : \limsup \text{area}(M_n \cap B(p, r)) = \infty \text{ for every } r > 0.\},$$

and suppose that Z lies in a closed region $N \subset \Omega$ with smooth connected mean-convex boundary ∂N , i.e., $g(H_{\partial N}, \xi) \geq 0$ on ∂N , where $H_{\partial N}$ is the mean curvature vector and ξ is the inward-pointing unit normal to ∂N . Then Z is a closed set and if $Z \cap \partial N \neq \emptyset$, then $Z \supset \partial N$.

Theorem 8.8 (White [16]). *Let Ω be an open subset in a Riemannian 3-manifold and g_n a sequence of smooth Riemannian metrics on Ω which converge smoothly to a metric g . Suppose that $M_n \subset \Omega$ is a sequence of properly embedded surfaces such that M_n is minimal with respect to g_n , and that the area and the genus of M_n are bounded independently of n . Then after passing to a subsequence, M_n converges to*

a smooth, properly embedded g -minimal surface M' . For each connected component Σ of M' , either

- (1) the convergence to Σ is smooth with multiplicity one, or
- (2) the convergence is smooth (with some multiplicity greater than 1) away from a discrete set S .

In the second case, if Σ is two-sided, then it must be stable.

We may now proceed.

Theorem 8.9. *In Case I, the A_n converge smoothly to $A \in \mathfrak{A}$, and $\Pi(A) = \Gamma \in \mathfrak{C}$.*

Proof. Using classical elliptic estimates and the Arzelà-Ascoli theorem, some subsequence of the u_n^\pm converge smoothly to functions u^\pm on $\mathbb{H}^2 \setminus B(q, R)$, the graphs of which are minimal. This means that the truncated minimal surfaces $M_n := A_n \cap (B(q, R) \times \mathbb{R})$ are annuli with smoothly converging boundaries $\partial M_n = \widehat{\gamma}_n^\pm$, so in particular, the lengths of ∂M_n are uniformly bounded.

The blowup set Z of the sequence A_n lies in the interior of the fixed solid cylinder. For h close to π , the catenoids C_h do not intersect $B(q, R) \times \mathbb{R}$, hence if the blowup set Z for this sequence were nonempty, then by decreasing h there would exist a point of first contact of some C_h with Z , which contradicts Theorem 8.7. Thus $Z = \emptyset$. Theorem 8.8 then implies that the A_n converge smoothly everywhere. \square

Remark 8.10. By an easy modification of this proof, Theorem 8.9 remains valid even if the limit curves (γ^-, γ^+) satisfy $\gamma^-(\theta) \leq \gamma^+(\theta)$ for all θ but $\gamma^- \not\equiv \gamma^+$.

Case II is more complicated. By [9, Theorem 4], the limiting boundary curves γ^\pm each span unique properly embedded minimal disks Y^\pm which are vertical graphs of functions v^\pm over all of \mathbb{H}^2 . In the following, we choose a sequence of horizontal dilations T_n such that $T_n(q_n)$ is a fixed point $q \in \mathbb{H}^2$. We normalize by assuming that the limit q_∞ of the q_n is the repulsive fixed point of T_n at infinity, and by rotation assume that $q_\infty = (1, 0)$. The attractive fixed point of T_n at infinity will be denoted Q_n , and for simplicity of notation, we assume that $Q_n \equiv (-1, 0)$ for all n (the case where Q_n is a converging sequence rather than a fixed point is essentially the same). We also denote by T_n the usual extension of these dilations to isometries of $\mathbb{H}^2 \times \mathbb{R}$.

Theorem 8.11. *In Case II, A_n converges smoothly on compact sets of $\mathbb{H}^2 \times \mathbb{R}$ to the union of minimal disks Y^\pm . The closures $\overline{A_n}$ converge as subsets of $\overline{\mathbb{H}^2} \times \mathbb{R}$ to the union of the vertical line segment $\{q_\infty\} \times [t^-, t^+]$ and the closures of Y^\pm . Here $(q_\infty, t^\pm) = ((1, 0), t^\pm)$ are points in γ^\pm , hence $t^\pm = \gamma^\pm(0)$, and thus necessarily in this case, $(\gamma^\pm)'(0) = 0$.*

Moreover, choosing T_n as above, the sequence $\overline{T_n(A_n)}$ converges to a vertical catenoid C_h smoothly in the interior and in $\mathcal{C}^{2,\alpha}$ on compact sets of $(\overline{\mathbb{H}^2} \setminus \{(-1, 0)\}) \times \mathbb{R}$. Since $h < \pi$, this convergence to a catenoid implies that $t^+ - t^- < \pi$.

Remark 8.12. In particular, if there is no pair of points $(q_\infty, t^\pm) \in Y^\pm$, one above the other, where the tangents are horizontal, then Case II cannot occur.

Proof. First note that, similarly to Case I, the ends E_n^\pm converge as minimal graphs, smoothly in the interior and in $\mathcal{C}^{2,\alpha}$ on compact sets of $(\overline{\mathbb{H}^2} \setminus \{q_\infty\}) \times \mathbb{R}$, to the minimal disks Y^\pm . On the other hand, the dilated boundary curves $\sigma_n^\pm = T_n(\gamma_n^\pm)$ converge in $\mathcal{C}^{2,\alpha}$ to the constant maps $\sigma^\pm(\theta) = t^\pm$ away from $\theta = \pi$, i.e., away from the point $(-1, 0)$. Hence $\overline{T_n(A_n)}$ converges in $\mathcal{C}^{2,\alpha}$ away from $\{(-1, 0)\} \times \mathbb{R}$.

We deduce from this that $T_n(A_n)$ converges to an embedded minimal annulus Σ . At first we only know that $\partial\Sigma$ consists of the two circles $\sigma^+ \sqcup \sigma^-$ and two (possibly overlapping) line segments $\{(-1, 0)\} \times J^\pm$. We claim that in fact $\partial\Sigma$ consists only of the two circles, so that by [10, Theorem 2.1], Σ equals a rotationally invariant catenoid C_h . To prove this, write $J^\pm = [a^\pm, b^\pm]$, and suppose that $b^+ > t^+$. By comparing with the family of minimal disks bounded by the circular arc $\sigma^+([- \pi + \epsilon, \pi - \epsilon])$, which has endpoints (p', t^+) , (p'', t^+) , together with the line segments $p' \times [t^+, b^+ + \epsilon]$, $p'' \times [t^+, b^+ + \epsilon]$ and $c \times \{b^+ + \epsilon\}$ where c is the arc $p'p''$, we see that $b^+ = t^+$. Similarly, $a^- = t^-$. This proves that Σ is entirely contained in the slab $\mathbb{H}^2 \times [t^-, t^+]$. We now appeal to Proposition 8.15 below for the fact that the limit set contains the entire segment $\{(-1, 0)\} \times [t^-, t^+]$.

At this stage, Σ is a minimal surface, the boundary at infinity of which is contained in the union of the two parallel circles $\sigma^+ \sqcup \sigma^-$ and the vertical segment $\{(-1, 0)\} \times [t^-, t^+]$. Now consider the geodesic η in \mathbb{H}^2 joining $(1, 0)$ to $(-1, 0)$ as well as the geodesic γ_s orthogonal to η and passing through the point $(s, 0)$, $-1 < s < 1$. We denote by P_s the vertical plane $\gamma_s \times \mathbb{R}$. Let \mathcal{U}_s and \mathcal{V}_s be the two connected components of $(\mathbb{H}^2 \times \mathbb{R}) \setminus P_s$, with $(1, 0, 0) \in \overline{\mathcal{U}_s}$. We write $\Sigma'_s := \Sigma \cap \overline{\mathcal{U}_s}$, $\Sigma''_s := \Sigma \cap \overline{\mathcal{V}_s}$ and Σ_s^* the reflection of Σ'_s with respect to P_s . Reasoning as in Proposition 3.3, we deduce Σ_s^* does not intersect Σ'_s except at the boundary when s is very close to 1. We claim that this is always the case.

If this were not the case, there would be a first point of interior contact, so that $\Sigma_s^* = \Sigma''_s$ for some s . But this is impossible since then the boundary of A would have another segment at $\{(1, 0)\} \times [t^-, t^+]$. This means that Σ_s^* does not intersect Σ'_s except at the boundary, for all s . In particular, Σ would be simply connected, which is absurd.

These arguments show that the boundary at infinity of the limit of $T_n(A_n)$ is the pair of parallel circles $\sigma^+ \sqcup \sigma^-$, and hence $T_n(A_n)$ converges to a catenoid C_h with axis $\{(0, 0)\} \times \mathbb{R}$.

The limit of the translated catenoids $T_n^{-1}(C_h)$ contains the entire line segment $\{(-1, 0)\} \times [t^-, t^+]$, hence the same must be true for the limit of the A_n .

It remains to show that the tangent lines to (the undilated curves) γ^\pm at (q_∞, t^\pm) are horizontal, i.e., that $(\gamma^\pm)'(0) = 0$. This relies on a flux calculation. We recall

from §4 that if $1/\kappa$ is the normal derivative of the graph function C_h at $r = 1$ and Z is any horizontal Killing field, then

$$(8.1) \quad \text{Flux}(C_h, \eta, E_3) = \frac{2\pi}{\kappa}, \quad \text{Flux}(C_h, \eta, Z) = 0.$$

Claim 8.13. *Parametrizing the (undilated) ends E_n^\pm by graph functions u_n^\pm , then as $n \rightarrow \infty$,*

$$(8.2) \quad \int_0^{2\pi} (u_n^+)_r(1, \theta) d\theta \rightarrow 2\pi/\kappa,$$

$$(8.3) \quad \int_0^{2\pi} (u_n^+)_r(1, \theta)(u_n^+)_\theta(1, \theta) d\theta \rightarrow 0.$$

Indeed, since $T_n(A_n) \rightarrow C_h$ smoothly on compact sets, there is a connected component λ_n of $T_n(A_n) \cap \{t = 0\}$ which generates the homology $H_1(T_n(A_n))$. Using the smooth convergence of $T_n(A_n)$ and the fact that fluxes are independent of the representative of homology class and invariant under isometries, then for each $\epsilon > 0$,

$$\left| \text{Flux}(T_n(A_n), \lambda_n, E_3) - \frac{2\pi}{\kappa} \right| < \epsilon, \quad |\text{Flux}(T_n(A_n), \lambda_n, Z)| < \epsilon.$$

for n sufficiently large. The claim follows.

Now focus just on the top curve, and for simplicity, drop the $+$ superscript. As noted earlier, γ_n bounds a unique minimal disk Y_n which is the vertical graph of a function $v_n(r, \theta) \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{H}^2})$, cf. [5, Proposition 3.1]. As $n \rightarrow \infty$, Y_n converges to a minimal disk Y with graph function v . The limit u of the functions u_n is attained in $\mathcal{C}^{2,\alpha}$ on $\overline{\mathbb{H}^2} \setminus \{(1, 0)\}$. Using Y_n as a barrier, we have

$$(8.4) \quad (v_n)_r(1, \theta) \leq (u_n)_r(1, \theta) \quad \forall \theta,$$

and since the v_n converge in $\mathcal{C}^{2,\alpha}(\overline{\mathbb{H}^2})$ to some function v , with graph Y , we obtain that

$$(8.5) \quad (u_n)_r(1, \theta) \geq -a > -\infty,$$

uniformly in θ and for all n . Moreover, since the pullbacks $T_n^*u_n = u_n \circ T_n$ converge along with their derivatives when $\theta \neq \pm\pi$, the radial derivatives of these functions are strictly positive for $|\theta| \leq \pi - \epsilon$ and n large. Since T_n is conformal, the radial derivatives of the original functions u_n are positive on $T_n^{-1}(\mathbb{S}^1 \setminus I)$ where I is any small interval around $(-1, 0)$.

Claim 8.14. *For each $\zeta > 0$ there exists a sequence of decreasing open arcs $\Upsilon_n \subset \mathbb{S}^1$ converging to $(1, 0)$ such that*

$$\int_{\Upsilon_n} (u_n)_r \geq \frac{2\pi}{\kappa} - \zeta.$$

To prove this, note that if this were to fail, then for some $\zeta > 0$ and any neighborhood \mathcal{V} of $(1, 0)$ in \mathbb{S}^1 , some subsequence, still labeled u_n , would satisfy

$$\int_{\mathcal{V}} (u_n)_r < \frac{2\pi}{\kappa} - \zeta.$$

Now, Y is simply connected, hence $\int_{\mathbb{S}^1} u_r = 0$, so given any $\zeta' > 0$, there exists a neighborhood \mathcal{V} of $(1, 0)$ in \mathbb{S}^1 such that

$$\left| \int_{\mathbb{S}^1 \setminus \mathcal{V}} u_r \right| < \zeta'.$$

But $(u_n)_r \rightarrow u_r$ uniformly on $\mathbb{S}^1 \setminus \mathcal{V}$, so

$$\left| \int_{\mathbb{S}^1 \setminus \mathcal{V}} (u_n)_r \right| < \zeta', \quad n \gg 1,$$

which implies that

$$\int_{\mathbb{S}^1} (u_n)_r = \int_{\mathbb{S}^1 \setminus \mathcal{V}} (u_n)_r + \int_{\mathcal{V}} (u_n)_r < \zeta' + 2\pi/\kappa - \zeta$$

for arbitrarily large n . But we can choose $\zeta' < \zeta$, which then contradicts (8.2). Hence the decreasing sequence of intervals \mathcal{V}_n with the stated properties exists.

Finally, since the integral of u_r over the entire circle vanishes, its integral over the complement \mathcal{U} of any sufficiently small neighborhood of $(1, 0)$ can be made arbitrarily small. Since $(u_n)_r \rightarrow u_r$ on \mathcal{U} , we may assume that the intersection of all the \mathcal{V}_n equals the single point $(1, 0)$. This finishes the proof of the claim.

We now show that $u_\theta(1, 0) = 0$. If this were not the case, then it is either positive or negative, and to be definite we assume that it is positive. Take an arc $\sigma \subset \mathbb{S}^1$ centered at $(1, 0)$ and positive constants $c_1 < c_2$ such that

$$0 < c_1 < u_\theta|_\sigma < c_2;$$

since $(u_n)_\theta \rightarrow u_\theta$ on σ , then for n large,

$$(8.6) \quad 0 < c_1 < (u_n)_\theta|_\sigma < c_2.$$

Now consider the horizontal flux

$$\text{Flux}(A_n, \lambda_n, Z) = \int_{\mathbb{S}^1} (u_n)_r(1, \theta)(u_n)_\theta(1, \theta) d\theta,$$

where Z is the horizontal Killing field Z induced by rotations. Fixing $\zeta > 0$, (8.3) implies that $|\text{Flux}(A_n, \lambda_n, Z)| < \zeta$ when n is large. In addition,

$$\int_{\mathbb{S}^1} u_r(1, \theta)u_\theta(1, \theta) d\theta = 0,$$

so there is an arc β of length less than ζ , with $(1, 0) \in \beta \subset \sigma$, and satisfying

$$(8.7) \quad \left| \int_{\mathbb{S}^1 \setminus \beta} u_r(1, \theta) u_\theta(1, \theta) d\theta \right| < \zeta.$$

Since $(u_n)_r(u_n)_\theta \rightarrow u_r u_\theta$ on $\mathbb{S}^1 \setminus \beta$, we see from (8.7) that for n large,

$$(8.8) \quad \left| \int_{\mathbb{S}^1 \setminus \beta} (u_n)_r(1, \theta) (u_n)_\theta(1, \theta) d\theta \right| < \zeta.$$

We may as well assume that $\Upsilon_n \subset \beta$. Then, recalling that $(u_n)_r > 0$ on Υ_n , and using (8.5), (8.6) and (8.8), we obtain

$$\begin{aligned} \zeta > \text{Flux}(A_n, \lambda_n, Z) &= \int_{\Upsilon_n} (u_n)_r(u_n)_\theta + \int_{\beta \setminus \Upsilon_n} (u_n)_r(u_n)_\theta + \int_{\mathbb{S}^1 \setminus \beta} (u_n)_r(u_n)_\theta \\ &\geq c_1 \int_{\Upsilon_n} (u_n)_r - c_2 a |\beta \setminus \Upsilon_n| - \zeta. \end{aligned}$$

Finally, by Claim 8.14 and the fact that $|\beta| < \zeta$, we see finally that

$$\zeta > \text{Flux}(A_n, \lambda_n, Z) \geq c_2(2\pi/\kappa - \zeta) + c_3\zeta,$$

which is a contradiction when ζ is small. This shows that $u_\theta(0) = 0$ and completes the proof of the Theorem. \square

Of course, Case II is not vacuous since for example $A_n = T_n(C_h)$ is a sequence which diverges in this manner.

Before proceeding to Case III we consider the limit set Λ_∞ in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ of the sequence A_n . Certainly $\Gamma^\pm \subset \Lambda_\infty$, and we set $\ell_\infty = \overline{\Lambda_\infty \setminus (\Gamma^+ \cup \Gamma^-)}$. We have shown that in Case I, $\ell_\infty = \emptyset$ and in Case II, ℓ_∞ is a single vertical segment of length less than π . We now study what can happen in the remaining case.

Proposition 8.15. *Let A_n be a sequence in \mathfrak{A} , with Λ_∞ and ℓ_∞ as above. Then ℓ_∞ is a union of vertical segments in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ joining Γ^+ , Γ^- .*

Proof. Suppose that there exists a point $(q_\infty, \tau_0) \in \ell_\infty$ such that there is a point $(q_\infty, \tau_1) \in \Gamma^+$ with $\tau_0 < \tau_1$ and $\{q_\infty\} \times (\tau_0, \tau_0 + \epsilon) \cap \ell_\infty = \emptyset$ for some $\epsilon > 0$. Choose a sequence $(q_n, t_n) \in A_n$ which converges to (q_∞, τ_0) . Now choose a sequence of horizontal isometries T_n so that $T_n((q_n, t_n)) = (\bar{q}, t_n)$ for some fixed point $\bar{q} \in \mathbb{H}^2$. The sequence of minimal annuli $T_n(A_n)$ converges to a varifold in $\mathbb{H}^2 \times \mathbb{R}$. (If there is no local area concentration, then A'_∞ is a properly embedded minimal surface, but it may in general have multiplicities.) Let A'_∞ denote the connected component of this varifold which contains (\bar{q}, τ_0) .

By construction, $A'_\infty \subset \mathbb{H}^2 \times (-\infty, \tau_0]$ and its ideal boundary lies in $\partial_\infty \mathbb{H}^2 \times (-\infty, \tau_0]$, and $(\bar{q}, \tau_0) \in A'_\infty$. This violates the general maximum principle of White

[14]. This proves that $\{q_\infty\} \times [\tau_0, \tau_1] \in \ell_\infty$. A similar argument applies to show that there are no gaps above Γ^- . \square

Lemma 8.16. *Let A_n be a sequence in \mathfrak{A} and suppose that the unit normals ν_n to A_n at some sequence of points $p_n \in A_n$ are horizontal. Then p_n cannot converge to a point in $\Gamma^+ \cup \Gamma^-$.*

Proof. The idea is very similar to that in the proof of Proposition 5.2. If $p_n = (q_n, t_n)$ converges to a point in Γ^+ , for example, then choose a sequence of horizontal isometries T_n so that $T_n(q_n, t_n) = (\bar{q}, t_n)$, where $t_n \rightarrow \bar{t}$. The sequence of minimal surfaces $T_n(A_n)$ converges as a point set to the support of a varifold A_∞ . (This is where the proof differs from the earlier one, since in that case we were taking a sequence of dilates of a fixed surface E , so the limit is necessarily a minimal surface.) By assumption, $A_\infty \subset \mathbb{H}^2 \times [\bar{t}, \infty)$ and its ideal boundary contains $\partial_\infty \mathbb{H}^2 \times \{\bar{t}\}$. Furthermore, $(\bar{q}, \bar{t}) \in A_\infty$. Invoking the general maximum principle [14] again, we see that $A_\infty = \mathbb{H}^2 \times \{\bar{t}\}$. But this contradicts the fact that A_∞ has a horizontal normal vector at this point. \square

We can finally proceed with the analysis of sequences of surfaces satisfying the conditions of Case III.

Theorem 8.17. *Let A_n be a sequence satisfying Case III. Then ℓ_∞ contains at least one vertical segment. Moreover, for any vertical segment $\{q_\infty\} \times (t_1, t_2) \subset \ell_\infty$, there exists a sequence of points $p_n \in A_n$ where the unit normal to A_n is horizontal, with p_n converging to $(q_\infty, \bar{t}) \in \{q_\infty\} \times (t_1, t_2)$.*

Finally, if T_n is a sequence of horizontal isometries mapping p_n to (q_∞, \bar{t}) , then $T_n(A_n)$ converges smoothly on compact sets of $\mathbb{H}^2 \times \mathbb{R}$ to a parabolic generalized catenoid, see [3, 4], which is a limit of vertical catenoids and is foliated by horizontal horocycles. In particular, $t_2 - t_1 = \pi$.

Proof. By hypothesis, we can assume there exists, for any $n \in \mathbb{N}$, a point p_n in $A_n \cap \partial(\mathbb{D}(c_n, R_n) \times \mathbb{R})$ with horizontal normal vector such that, after passing to a subsequence, $\{p_n\}_{n \in \mathbb{N}}$ diverges to a point in $\{c_\infty\} \times \mathbb{R}$, for some $c_\infty \in \partial_\infty \mathbb{H}^2$. By Proposition 8.15 and Lemma 8.16, we get that ℓ_∞ contains at least the vertical segment $\{c_\infty\} \times (t^-, t^+)$, being $q^\pm = (c_\infty, t^\pm) \in \Gamma^\pm$.

Let us now prove that $\Sigma_n = T_n(A_n)$ converge to a generalized catenoid foliated by horizontal horocycles (Claim 8.19). We call $\sigma^\pm = \partial_\infty \mathbb{H}^2 \times \{t^\pm\}$. Thus we have that the asymptotic boundary curves $\partial_\infty \Sigma_n$ converge to $\sigma^+ \cup \sigma^-$.

Since $R_n \rightarrow \infty$, the disks $\mathcal{H}_n = T_n(\mathbb{D}_{\mathbb{H}^2}(c_n, R_n))$ converge to the horodisk \mathcal{H}_∞ at $-c_\infty$ passing through the origin. We can consider a larger horodisk $\tilde{\mathcal{H}}_\infty$ at $-c_\infty$ containing \mathcal{H}_∞ . It is clear that $\tilde{\mathcal{H}}_\infty$ also contains any \mathcal{H}_n , for n big enough.

Let v_n^\pm be the smooth function defined on $\Lambda := \mathbb{H}^2 \setminus \tilde{\mathcal{H}}_\infty$ whose graph represents the ends of Σ_n around σ^\pm . By the Arzelà-Ascoli theorem, a subsequence of $\{v_n^\pm\}_{n \in \mathbb{N}}$

converges uniformly on compact subsets of Λ to a minimal graph v_∞^\pm , with $v_\infty^\pm = t^\pm$ on $\partial_\infty \mathbb{H}^2 \setminus \{-c_\infty\}$.

Now set $M_n := \Sigma_n \cap (\tilde{\mathcal{H}}_\infty \times \mathbb{R})$; each M_n is an annulus bounded by two curves ϱ_n^\pm in $\partial \tilde{\mathcal{H}}_\infty \times \mathbb{R}$. By possibly enlarging $\tilde{\mathcal{H}}_\infty$, we can assume that $\{\varrho_n^\pm\}_{n \in \mathbb{N}}$ converges uniformly to the graph of $v_\infty^\pm|_{\partial \tilde{\mathcal{H}}_\infty}$. Hence it is easy to see that the boundary measures of the annuli M_n are uniformly bounded on compact sets. Thus, by Theorem 8.7, the area blowup set Z of the minimal annuli M_n (or Σ_n), which lies in the solid horocylinder, obeys the same maximum principles that hold for properly embedded minimal surfaces without boundary.

Let \mathcal{D} be a parabolic generalized catenoid foliated by horocycles at points in $\{-c_\infty\} \times \mathbb{R}$. Up to a vertical translation, we can assume that \mathcal{D} is a bigraph symmetric with respect to $\mathbb{H}^2 \times \{0\}$, and its asymptotic boundary consists of $(\partial_\infty \mathbb{H}^2 \times \{\pm\pi/2\}) \cup (\{-c_\infty\} \times [-\pi/2, \pi/2])$. Consider a dilation T from $-c_\infty$ such that $T(\mathcal{D})$ is disjoint from $\tilde{\mathcal{H}}_\infty \times \mathbb{R}$. Now we start dilating $T(\mathcal{D})$ towards $-c_\infty$, converging to $(\{-c_\infty\} \times [-\pi/2, \pi/2])$. By the maximum principle (Theorem 8.7) we get that none of these copies of \mathcal{D} can touch the area blowup set Z . We repeat this argument for vertically translated copies of \mathcal{D} , obtaining that Z is empty.

Since in $\Lambda \times \mathbb{R}$ the convergence of the annuli Σ_n is smooth with multiplicity one, then we can apply Theorem 8.8 to deduce that we have the same convergence inside of the solid horocylinder. Therefore, the minimal annuli Σ_n converge to a complete, embedded minimal surface Σ_∞ with asymptotic boundary $\sigma^+ \cup \sigma^-$ and possibly some points in $\{-c_\infty\} \times (t^-, t^+)$. We are going to prove that Σ_∞ coincides with a isometric copy of \mathcal{D} . Firstly, let us prove that their asymptotic boundaries have the same behavior.

Claim 8.18. $\partial_\infty \Sigma_\infty = \sigma^+ \cup \sigma^- \cup (\{-c_\infty\} \times (t^-, t^+))$.

By hypothesis, there exist a point \hat{p}_n in A_n with horizontal normal vector, such that the distance in \mathbb{H}^2 between the vertical projections of the points p_n and \hat{p}_n over \mathbb{H}^2 diverges to $+\infty$. In particular, the asymptotic boundary of Σ_∞ contains at least one point in $\{-c_\infty\} \times (t^-, t^+)$. Claim 8.18 follows from Proposition 8.15.

Up to a vertical translation, we can assume $t^- = -t^+$.

Claim 8.19. $t^+ = \pi/2$ and $\Sigma_\infty = \mathcal{D}$, up to an isometry.

Suppose first that $t^+ > \pi/2$. Consider a parabolic generalized catenoid \mathcal{E} of height $2t_0 \in (\pi, 2t^+)$, obtained in [3, 4, 12] (see also [6, 7]), foliated by horizontal equidistant curves to a fixed vertical plane. \mathcal{E} is a minimal disk whose boundary consists of an arc c in $\partial \mathbb{H}^2 \times \{t_0\}$, $\partial \mathbb{H}^2 \times \{-t_0\}$ and the corresponding vertical segments in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ joining the endpoints of such horizontal arcs. \mathcal{E} is invariant by dilations in the direction of the complete geodesic joining the endpoints of c . We can assume that c is symmetric with respect to $\{c_\infty\} \times \mathbb{R}$ and it is short enough

(in the Euclidean metric) so that \mathcal{E} does not intersect Σ_∞ (this is possible since we know that $\Sigma_\infty \cap (\Lambda \times \mathbb{R})$ is the union of two disjoint vertical graphs). Considering dilated copies of \mathcal{E} from c_∞ , we get an intersection point with Σ_∞ . That contradicts the maximum principle and proves $t^+ \leq \pi/2$.

Let Γ denote the complete geodesic from c_∞ to $-c_\infty$, and let Υ be a component of $\mathbb{H}^2 \setminus \Gamma$. We call $\Sigma'_\infty = \Sigma_\infty \cap (\Upsilon \times \mathbb{R})$. By the maximum principle, $\partial\Sigma'_\infty$ cannot have bounded components and, by Claim 8.18, the asymptotic boundary of Σ'_∞ is

$$\partial_\infty \Sigma'_\infty = ((\sigma^+ \cup \sigma^-) \cap (\partial_\infty \Upsilon \times \mathbb{R})) \cup (\{-c_\infty\} \times (-t^+, t^+)).$$

Hence Σ'_∞ is a disk whose boundary $\partial\Sigma'_\infty$ is a curve contained in $\Sigma_\infty \cap (\Gamma \times \mathbb{R})$ joining Γ^+ to Γ^- . (Again we are using that Σ_∞ is embedded and that $\Sigma_\infty \cap (\Lambda \times \mathbb{R})$ is the union of two disjoint vertical graphs with asymptotic boundary Γ^-, Γ^+ .)

We take a rotated copy of a parabolic generalized catenoid, \mathcal{D}' , bounded by $\partial_\infty \mathbb{H}^2 \times \{\pm\pi/2\}$ and a vertical segment over a point b_∞ in $\partial_\infty \mathbb{H}^2 \setminus \overline{\partial_\infty \Upsilon}$. We can consider a dilation from $-b_\infty$ so that \mathcal{D}' does not intersect Σ'_∞ . Now consider the continuous family of rotations \mathcal{R}_l around the origin mapping b_∞ to the points from b_∞ to $-c_\infty$ along $\partial_\infty \mathbb{H}^2 \setminus \partial_\infty \Upsilon$. By the maximum principle using the family $\{\mathcal{R}_l(\mathcal{D})\}_l$, going from \mathcal{D}' to \mathcal{D} , we get that Σ'_∞ is contained in “the exterior” of \mathcal{D} , i.e. in the component of $(\mathbb{H}^2 \times \mathbb{R}) \setminus \mathcal{D}$ which contains $(\partial_\infty \mathbb{H}^2 \setminus \{-c_\infty\}) \times \{0\}$ in its asymptotic boundary. By a symmetric argument, we get that Σ_∞ lies in “the exterior” of \mathcal{D} . Now, consider dilated copies of \mathcal{D} from $-c_\infty$. By the maximum principle, Claim 8.19 follows.

To finish the proof of Theorem 8.17, it remains to prove that, for any vertical segment in ℓ_∞ , there exists $\{p'_n\}_{n \in \mathbb{N}}$ converging to a point in this vertical segment, there $p'_n \in A_n$ is a point with horizontal normal vector.

Let us consider $\{c'_\infty\} \times (\hat{t}^-, \hat{t}^+) \in \ell_\infty$, with $\hat{q}^- = (c'_\infty, \hat{t}^-) \in \Gamma^-$ and $\hat{q}^+ = (c'_\infty, \hat{t}^+) \in \Gamma^+$. For any $n \in \mathbb{N}$, let α_n be a curve contained in A_n joining \hat{q}^- , \hat{q}^+ and converging to $\{c'_\infty\} \times (\hat{t}^-, \hat{t}^+)$. Since A_n separates $\mathbb{H}^2 \times \mathbb{R}$, then we can assume that the unit normal vector to A_n points up in a neighborhood of \hat{q}^- and points down in a neighborhood of \hat{q}^+ . In particular, there must be a point $p'_n \in \alpha_n$ with horizontal normal vector. \square

The previous theorem has a useful application for rotationally invariant annuli.

Theorem 8.20. *Let $m \in \mathbb{N}$, $m \geq 2$ and consider a sequence of minimal annuli $\{A_n\}_{n \in \mathbb{N}}$ in \mathfrak{A}_m . Assume that the sequence of boundary curves $\{\Gamma_n := \Pi(A_n)\}_{n \in \mathbb{N}}$ (which is a sequence of curves in \mathfrak{C}_m) satisfies that $\{\Gamma_n\}_{n \in \mathbb{N}} \rightarrow \Gamma_0 \in \mathfrak{C}_m$ in the $\mathcal{C}^{2,\alpha}$ topology. Then, up to a subsequence, $\{A_n\}_{n \in \mathbb{N}}$ converges (smoothly on compact sets) to a properly embedded minimal annulus $A_0 \in \mathfrak{A}_m$ such that $\Pi(A_0) = \Gamma_0$.*

Proof. Since the annulus A_n is in \mathfrak{A}_m , we deduce the existence of a radius $R_n > 0$ such that

$$A_n \setminus \mathbb{D}_{\mathbb{H}^2}(R_n) \times \mathbb{R}$$

is the union of two vertical graphs. Theorem 8.17 says us that the sequence $\{R_n\}_{n \in \mathbb{N}}$ is bounded; otherwise the limit curve Γ_0 cannot belong to \mathfrak{C}_m , because there must be point whose vertical distance is precisely π .

We can now reason as in the proof of Theorem 8.9 to deduce the existence of the limit annulus A_0 . As A_n is \mathcal{R}_m -invariant, for all $n \in \mathbb{N}$, then the limit is also \mathcal{R}_m -invariant. \square

Corollary 8.21. *Given $m \in \mathbb{N}$, $m \geq 2$, then the projection $\Pi : \mathfrak{A}_m \rightarrow \mathfrak{C}_m$ is proper.*

We finish this section with an observation similar to Remark 8.10.

Remark 8.22. If the limit curve $\Gamma_0 \equiv (\gamma_0^+, \gamma_0^-)$ in Theorem 8.20 satisfies $\gamma_0^+(\theta) \geq \gamma_0^-(\theta)$, for all $\theta \in \mathbb{S}^1$, but $\gamma_0^+ \neq \gamma_0^-$, then the statement of the theorem remains true.

9. THE ASYMPTOTIC PLATEAU PROBLEM FOR MINIMAL ANNULI

We now assemble the results above to prove various types of local and global existence theorems. Our goal, of course, is to determine as much information as possible about the space of minimally fillable curves in \mathfrak{C} . We start with a few qualitative remarks. First, it is apparent that the projection $\Pi : \mathfrak{A} \rightarrow \mathfrak{C}$ has some sort of fold around the catenoid family. Indeed, $\Pi^{-1}(\mathbb{S}^1 \times \{\pm h\})$ is noncompact for every $0 < h < \pi/2$. In addition, we have exhibited a specific infinite dimensional family of curves converging to a pair of circles which are not minimally fillable, while on the other hand, because of the existence of nondegenerate minimal annuli arbitrarily near the catenoid, any one of these pairs of parallel circles is the limit of pairs of curves which are in the interior of the image of Π . Thus a precise characterization of this image may not be possible. We present two separate existence results which are nonperturbative and give the existence of infinite-dimensional families of minimal annuli far away from the catenoid. The key question not answered here is whether there is indeed a failure of compactness, or equivalently, if Π is proper away from the catenoid family. We have reason to suspect that there are many other regions where properness may fail, but have so far been unsuccessful in demonstrating this.

The most general existence result that we can prove is the following theorem, which summarizes all the information that we have about the map $\tilde{\Pi}$.

Theorem 9.1. *The map $\tilde{\Pi} : \tilde{\mathfrak{A}}^* \rightarrow \mathcal{C}^{2,\alpha}(\mathbb{S}^1)^2 \times \mathbb{R} \times \mathbb{D}$ is a proper Fredholm map of index 0 and degree 1. In particular, given any $\gamma^\pm \in \mathcal{C}^{2,\alpha}(\mathbb{S}^1)^2$, there exist constants a_0, a_1, a_2 so that the pair $(\gamma^+ + a_0 + a_1 \cos \theta + a_2 \sin \theta, \gamma^-)$ bounds a proper, Alexandrov-embedded, minimal annulus.*

The proof of this theorem is a direct consequence of Theorem 7.4, Proposition 7.7 and Theorem 8.5.

Theorem 9.1 asserts that we can prescribe the bottom curve of an Alexandrov-embedded minimal annulus, as well as the top curve up to a translation and tilt, and in addition the “center of the neck” and certain fluxes.

9.1. Solutions with symmetry. Let G be any finite group of isometries of $\mathbb{H}^2 \times \mathbb{R}$ which leaves invariant some fixed catenoid A_0 . Assume in addition that no element of $\mathfrak{J}^0(A_0)$ is left invariant by G . There are two main examples: the group \mathcal{R}_k , $k \geq 2$, generated by the rotation by angle $2\pi/k$ around the vertical axis which is the line of symmetry of the catenoid, and the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by reflection across a vertical plane bisecting the catenoid and rotation by π around the line orthogonal to that plane which intersects the midpoint of the catenoid’s neck. Examples of curves with the first type of symmetry are obvious. For the second, a key example is a pair of parallel ellipses.

Theorem 9.2. *Let $\Gamma = \gamma^\pm$ be any G -invariant pair of curves such that*

$$(9.1) \quad \sup_{\theta} |\gamma^+(\theta) - \gamma^-(\theta)| < \pi.$$

Then Γ is minimally fillable.

Proof. We shall work in the setting of G -invariant objects, mappings, etc. In this context, the catenoid A_0 is nondegenerate and the local deformation theorem is an immediate consequence of the implicit function theorem. Denoting by \mathfrak{A}_G and \mathfrak{C}_G the Banach manifolds of G -invariant minimal annuli and boundary curves, we have that $\Pi_G : \mathfrak{A}_G \rightarrow \mathfrak{C}_G$ is Fredholm of degree 0, just as in the non- G -invariant setting. Furthermore, by the compactness arguments of the last section, this mapping is proper over the space of elements of \mathfrak{C}_G satisfying (9.1). We do not need the full set of arguments developed in the last section. Indeed, for G -invariant sequences of minimal annuli A_i , it is necessary to rule out that the neck shrinks or expands, but not that it remains of bounded size and escapes to infinity, since that is ruled out by G -invariance. On the other hand, we do require non- G -equivariant techniques to rule out the possibility that the necksize increases without bound.

We have shown that Π_G is a proper Fredholm map of index 0. It therefore has a \mathbb{Z} -valued degree, defined by the formula

$$\deg(\Pi_G) = \sum_{A \in \Pi_G^{-1}(\Gamma)} (-1)^{\text{index}_G(A)},$$

where Γ is any regular value of Π_G . By the Sard-Smale theorem, a generic element of \mathfrak{C}_G is regular, and since we have shown that there exists a neighborhood in \mathfrak{A}_G around the catenoid which projects diffeomorphically to a neighborhood in \mathfrak{C}_G , there exists a regular value Γ for which $\Pi_G^{-1}(\Gamma)$ is nonempty. Recall that $\text{index}_G(A)$ is the

number of negative eigenvalues, of the (negative of the) Jacobi operator acting on G -invariant functions.

It remains therefore to prove that the degree of Π_G is nonzero. However, the pair of parallel circles Γ_0 separated by $h < \pi$ bounds only the catenoid, so the sum above has only one term, which means that $\deg(\Pi_G)$ is equal to either 1 or -1 . In any case it is nonzero. This proves the G -invariant existence result. \square

We single out one particularly interesting family of solutions. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, acting as described above, and let Γ_s denote a family of parallel ellipses, the upper one a translate by some fixed amount $h < \pi$ of the lower. The parameter s measures the tilt, and varies between $s = 0$ (where Γ_0 is just the pair of parallel horizontal circles) to the extreme limit where these ellipses become more and more vertical.

9.2. Solutions with admissible boundary. Our other existence result allows us to consider more general curves and annuli.

9.2.1. Properness over the space of admissible curves. The compactness results that we got in Section 8 motivate the following

Definition 9.3. Let $\Gamma = (\gamma^+, \gamma^-)$ be a curve in \mathfrak{C}^π . We say that Γ is **admissible** if $\frac{d}{d\theta}(\gamma^+(\theta), \gamma^-(\theta)) \neq (0, 0)$, for all $\theta \in [0, 2\pi)$, and set $\Omega := \{\Gamma \in \mathfrak{C}^\pi : \Gamma \text{ is admissible}\}$. This is an **open** subset of \mathfrak{C} . We also write $\mathcal{W} := \Pi^{-1}(\Omega)$.

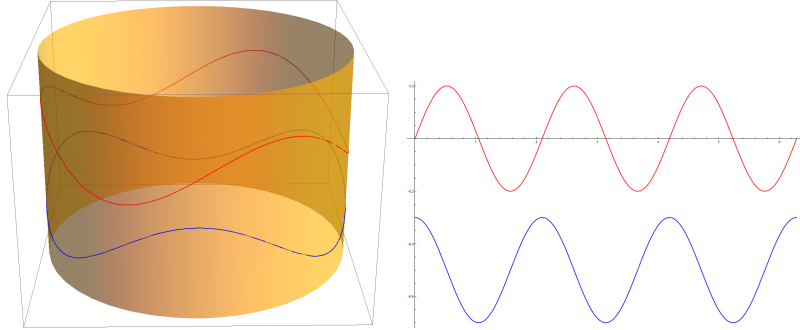


FIGURE 8. An admissible curve at $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$.

Taking the previous definition into account, we obtain the following compactness results:

Theorem 9.4 (Compactness). *Let A_n be a sequence in \mathfrak{A} such that $\Pi(A_n)$ converges to $\Gamma_0 \in \Omega$. Then, up to a subsequence, A_n converges to a minimal annulus $A_0 \in \mathfrak{A}^\pi$.*

Proof. Since $\Gamma_0 \in \Omega$, Theorem 8.11 and Theorem 8.17 imply that A_n is in Case I in Section 8.1, so Theorem 8.9 concludes the proof. \square

This has an immediate consequence.

Corollary 9.5. *The map $\Pi|_{\mathcal{W}} : \mathcal{W} \rightarrow \Omega$ is **proper**.*

If Γ is admissible, then

$$\alpha_\Gamma(\theta) = \frac{d}{d\theta} (\gamma^+(\theta), \gamma^-(\theta)) : \mathbb{S}^1 \rightarrow \mathbb{C} \setminus \{0\}$$

is a smooth loop. and hence homotopic in $\mathbb{C} \setminus \{0\}$ to a standard n -cycle, $\alpha_n(\theta) = e^{in\theta}$. We say that Γ is **n-admissible**. Given n -admissible curves Γ_0 and Γ_1 , there exists a smooth isotopy Γ_t between these amongst n -admissible curves.

Remark 9.6. Ω has a countable number of path-connected components

$$\Omega_n := \{\Gamma \in \Omega : \Gamma \text{ is } n\text{-admissible}\}.$$

Defining $\mathcal{W}_n := \Pi^{-1}(\Omega_n) \subset \mathcal{W}$, then the family of catenoids $\{C_h : h \in (0, \pi/2)\}$, is in the boundary of \mathcal{W}_n for every $n \in \mathbb{Z}$.

9.3. Existence results. Consider, for any $m \in \mathbb{N}$, $m \geq 2$, the rotation R_m , and the finite group \mathcal{R}_m it generates.

Fixing $h < \pi/2$, then $\Gamma_h = \mathbb{S}^1 \times \{-h, h\} \in \mathfrak{C}^\pi$ spans a unique centered catenoid C_h .

Proposition 9.7. *There exists an open neighborhood of Γ_h , $U \subset \mathfrak{C}_m$, such that for any $\Gamma \in U$, then there is a **unique** annulus $A \in \mathfrak{A}_m$ with $\Pi(A) = \partial A = \Gamma$.*

Proof. First notice that all the arguments in Section 6 remain true restricted to the space of \mathcal{R}_m -invariant functions over C_h .

By Theorem 2.2, the space of decaying Jacobi fields \mathfrak{J}_h^0 is generated by

$$\varphi = \frac{1-r^2}{r} \cos \theta, \quad \psi = \frac{1-r^2}{r} \sin \theta,$$

so $\text{Ker} \left(D \left(\Pi|_{\mathfrak{A}_m} \right) \Big|_{C_h} \right) = \{0\}$. The Inverse Function Theorem gives neighborhoods $C_h \ni W \subset \mathfrak{A}_m$ and $\Gamma_h \ni U \subset \mathfrak{C}_m$ such that $\Pi|_W : W \rightarrow U$ is a diffeomorphism.

To prove the uniqueness, we proceed by contradiction. Assume there exists a sequence $\Gamma_n \in \mathfrak{C}_m$ with $\Gamma_n \rightarrow \Gamma_h$ and such that there are two distinct \mathcal{R}_m -invariant minimal annuli $A_n^1 \neq A_n^2$, each satisfying $\Pi(A_n^i) = \Gamma_n$. By Theorem 8.20, $A_n^1 \rightarrow C_h$ and $A_n^2 \rightarrow C_h$.

Write A_n^i as a normal graph over C_h of a function u_n^i , $i = 1, 2$. Define

$$v_n := \left(\frac{u_n^2 - u_n^1}{\|u_n^2 - u_n^1\|_\infty} \right).$$

This vanishes on ∂C_h since $(u_n^2 - u_n^1)|_{\Gamma_h} = 0$,

Choose $p_n \in C_h$ so that $v_n(p_n) = 1$. If p_n diverges in $\mathbb{H}^2 \times \mathbb{R}$, then take a horizontal translation T_n such that $T_n(p_n) = (0, t_n)$. Clearly $\{t_n\} \rightarrow \{\pm h\}$, so we choose a subsequence for which p_n converges to a point in $\mathbb{S}^1 \times \{h\}$. Then $v_n \circ T_n^{-1}$ converges to a Jacobi field on $\mathbb{H}^2 \times \{h\}$ which reaches a maximum at $(0, h)$. This is impossible. This proves that, up to a subsequence, $p_n \rightarrow p_0 \in C_h$.

It is now standard to deduce the existence of a limit $v := \lim_{n \rightarrow \infty} v_n$, which is a nontrivial \mathcal{R}_m -invariant element of $\mathfrak{J}^0(C_h)$. However, there are no such elements. This is a contradiction, and hence we have proved the uniqueness. \square

Remark 9.8. Reasoning as above, we can also prove that if A is \mathcal{R}_m -invariant and sufficiently close to C_h , then there are no \mathcal{R}_m -invariant elements of $\mathfrak{J}^0(A)$.

When $m = 2$, even more is true. In the following, U denotes the neighborhood of Γ_h provided by Proposition 9.7 for $m = 2$.

Proposition 9.9. *For every neighborhood $U' \subset U \subset \mathfrak{C}_2$ containing Γ_h , and for any even integer $n \neq 0$, there exists $\Gamma \in U' \cap \Omega_n$ such that the unique annulus $A \in \mathfrak{A}_2 \cap \mathcal{W}_n$ with $\Pi(A) = \Gamma$ is **non-degenerate**, in the sense that $\mathfrak{J}^0(A) = \{0\}$.*

Proof. Since n is even, we can construct a family $\{A_\varepsilon\}$ of minimal annuli satisfying, for all $|\varepsilon| < \varepsilon_0$:

- (a) $A_0 = C_h$ and $\Pi(A_\varepsilon) \subset U'$;
- (b) $A_\varepsilon \in \mathcal{W}_n$;
- (c) A_ε is \mathcal{R}_2 -invariant, but **not** \mathcal{R}_{2k} -invariant, $k > 1$.

By Proposition 10.1 below (see also Remark 10.2), we deduce that A_ε is non-degenerate, for almost all $\varepsilon > 0$. \square

Theorem 9.10. *If n is even and nonzero, the projection*

$$\Pi : \mathcal{W}_n \longrightarrow \Omega_n$$

is a proper map of degree $\pm 1 \pmod{2}$. In particular, given $\Gamma \in \Omega_n$, there exists a properly embedded minimal annulus such that $\Pi(A) = \Gamma$.

Proof. By Corollary 9.5, $\Pi|_{\mathcal{W}_n}$ is proper. Thus $\Pi|_{\mathcal{W}_n}$ has a well-defined degree:

$$\deg(\Pi|_{\mathcal{W}_n}) := \sum_{A \in \Pi^{-1}(\Gamma)} (-1)^{\text{index}(A)},$$

where Γ is any regular value of Π . Regular values are generic in \mathfrak{C} .

The rotation R_2 is a diffeomorphism of \mathcal{W}_n . Fix Γ_0 in the open neighborhood \mathcal{U} of Proposition 9.7, for $m = 2$. Then Γ_0 spans a unique R_2 -invariant annulus A_0 . By Proposition 9.9 we may choose Γ_0 so that A_0 is non-degenerate.

Enumerate the other (non-congruent) annuli in $\Pi^{-1}(\Gamma_0)$ by A_1, \dots, A_k . If Γ_0 is sufficiently close to $\mathbb{S}^1 \times \{-h, h\}$, then any non-symmetric solution A_i , $i \in \{1, \dots, k\}$, creates 2 different annuli $A_i^j := R_2^j(A_i)$, $j = 0, 1$, each in $\Pi^{-1}(\Gamma_0)$.

Let \mathfrak{U}_0 be an $R - 2$ -invariant neighborhood of A_0 in \mathcal{W}_n for which $\mathfrak{J}^0(A) = \{0\}$, for all $A \in \mathfrak{U}_0$. Choose further neighborhoods \mathfrak{U}_i of A_i in \mathcal{W}_n , $i = 1, \dots, k$ which are pairwise disjoint from each other and from \mathfrak{U}_0 , and such that $R_2(\mathfrak{U}_i) \cap \mathfrak{U}_j = \emptyset$, for all $i, j = 1, \dots, k$. Now choose $\Gamma \in \Omega_n$ near Γ_0 , which is a regular value of Π (possibly $\Gamma_0 = \Gamma$.) Our compactness results imply that

$$\Pi^{-1}(\Gamma) \subset \bigcup_{i=0}^k (\mathfrak{U}_i \cup R_2(\mathfrak{U}_i)).$$

Furthermore, clearly $\deg(\Pi|_{\mathfrak{U}_i}) = \deg(\Pi|_{R_2(\mathfrak{U}_i)})$, $i = 1, \dots, k$. Therefore

$$\deg(\Pi) = (-1)^{\text{index}(A)} + 2 \cdot \left(\sum_{i=1}^k \deg(\Pi|_{\mathfrak{U}_i}) \right),$$

where A is the unique element in $\Pi^{-1}(\Gamma) \cap \mathfrak{U}_0$. This concludes the proof. \square

9.4. The asymptotic Dirichlet problem for non-disjoint curves. In the preceding we have considered pairs of curves (γ^+, γ^-) satisfying $\gamma^+(\theta) > \gamma^-(\theta)$ for all θ . However, we can extend these results about R_m -invariant solutions to allow boundary curves $\Gamma = (\gamma^+, \gamma^-)$ for which $\gamma^+(\theta) \geq \gamma^-(\theta)$ for all θ , but $\gamma^+ \not\equiv \gamma^-$. With this sense of $\gamma_+ \geq \gamma_-$, define

$$\mathfrak{C}^* := \{(\gamma^+, \gamma^-) : \gamma^+ \geq \gamma^- \text{ and } \sup_{\theta \in \mathbb{S}^1} (\gamma^+(\theta) - \gamma^-(\theta)) < \pi\}$$

$$\mathfrak{C}_m^* := \{(\gamma^+, \gamma^-) \in \mathfrak{C}^* : (\gamma^+, \gamma^-) \text{ are } R_m\text{-invariant}\}$$

Theorem 9.11. *If $\Gamma^* \in \mathfrak{C}_m^*$, then there exists a complete, properly embedded minimal annulus A^* such that $\Pi(A^*) = \Gamma^*$.*

Proof. Consider a sequence of curves $\Gamma_n \in \mathfrak{C}_m$ converging to Γ^* . By Theorem 9.2, for each n there exists a properly embedded minimal annulus $A_n \in \mathfrak{A}_m$ such that $\Pi(A_n) = \Gamma_n$. Now use Theorem 8.20 and Remark 8.22 to deduce that a subsequence of A_n converges, smoothly on compact sets, to a minimal properly embedded minimal annulus A^* . By construction, this annulus satisfies $\Pi(A^*) = \Gamma^*$. \square

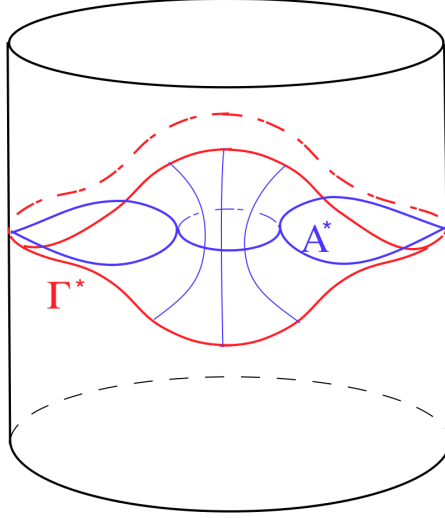


FIGURE 9. Annuli of this kind can be obtained as limits of our examples.

10. EXISTENCE OF NONDEGENERATE MINIMAL ANNULI

The main result of the previous section is of a somewhat general nature and does not preclude, for example, the possibility that *every* $A \in \mathfrak{A}$ is degenerate, i.e., that it is possible that $\mathfrak{J}^0(A) \neq 0$ for all $A \in \mathfrak{A}$. We prove here that this is not the case.

Proposition 10.1. *There exist minimal annuli arbitrarily near to any catenoid A_0 which are nondegenerate.*

Proof. We have proved that \mathfrak{A} is a smooth Banach manifold, so it makes sense to talk about smooth curves A_ϵ of minimal annuli which are deformations of the catenoid A_0 . These are normal graphs over A_0 of a smooth family of functions u_ϵ , and we set $\psi \in \mathfrak{J}(A)$ to be the ϵ -derivative of u_ϵ at $\epsilon = 0$. Thus $\psi \in \mathfrak{J}$, so by the results in the previous section, its leading coefficients are orthogonal to the normal derivatives of every $\phi \in \mathfrak{J}^0$. In other words,

$$(10.1) \quad (\psi_0^+, \psi_0^-) \perp (\cos \theta, -\cos \theta), \quad (\psi_0^+, \psi_0^-) \perp (\sin \theta, -\sin \theta).$$

Moreover, we can construct such families of minimal annuli for any Jacobi field which satisfies (10.1).

Now, within the space of Jacobi fields $\mathfrak{J}(A_\epsilon)$, there are two distinguished subspaces: the decaying Jacobi fields $\mathfrak{J}^0(A_\epsilon)$ and another space $\mathcal{D}(A_\epsilon)$ consisting of Jacobi fields on A_ϵ generated by horizontal dilations in $\mathbb{H}^2 \times \mathbb{R}$. We are assuming that $\mathfrak{J}^0(A_\epsilon)$ is nontrivial, and standard eigenvalue perturbation theory implies that $\dim \mathfrak{J}^0(A_\epsilon) \leq 2$. Furthermore, we also have that $\mathcal{D}(A_\epsilon)$ converges to $\mathfrak{J}^0(A_0)$ since

the latter is also generated by dilations. Thus when $\epsilon \neq 0$, the two spaces are quite close to one another. We are interested in the way in which one approaches the other.

Let us first assume that $\dim \mathfrak{J}^0(A_\epsilon) = 2$ for all small ϵ . We handle the other case later. Choose a codimension two subspace $W_\epsilon \subset \mathfrak{J}(A_\epsilon)$ which varies smoothly in ϵ and which is always complementary to $\mathfrak{J}^0(A_\epsilon)$ (for example, we can take an orthogonal complement with respect to some weighted Hilbert structure). Choose any smooth family $\phi_\epsilon \in \mathcal{D}(A_\epsilon)$, and decompose it into components in each of these subspaces, $\phi_\epsilon = \psi_\epsilon + w_\epsilon$, where $\psi_\epsilon \in \mathfrak{J}^0(A_\epsilon)$ and $w_\epsilon \in W_\epsilon$. Note that w_ϵ and ϕ_ϵ agree at ∂A_ϵ .

We now observe that the map which assigns to any $w \in W_\epsilon$ its leading coefficients, i.e., boundary values, (w_0^+, w_0^-) , has image equal to a codimension 2 subspace of $(\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2$ which depends smoothly on ϵ and which equals the subspace given by the orthogonality conditions (10.1) when $\epsilon = 0$. This map is, by definition, injective and also surjective. Hence by the open mapping theorem, the norm of any $w \in W_\epsilon$ is equivalent to the norm of its boundary values. This means that the rescaled sequence of functions

$$\tilde{w}_\epsilon = w_\epsilon / \sup_{\partial A_\epsilon} |w_\epsilon|$$

is uniformly bounded, independently of ϵ , and always attains the value 1 at some point of the boundary. Thus it has a well defined limit, \tilde{w} , which is therefore an element of $\mathfrak{J}(A_0)$. Its boundary values must satisfy (10.1).

Now let us compute the boundary values of the original Jacobi field ϕ_ϵ . Parametrizing the boundary curves of A_ϵ by the functions $f_\epsilon^\pm(\theta)$, suppose that D_λ is a family of horizontal dilations and let $A_{\epsilon,\lambda} = D_\lambda(A_\epsilon)$, corresponding in turn to a family of normal graphs over A_ϵ by a family of functions $u_{\epsilon,\lambda}$. We calculate readily that at $r = 1$ and at the upper and lower halves, up to an overall constant factor,

$$\left. \frac{d}{d\lambda} u_{\epsilon,\lambda}(1, \theta) \right|_{\lambda=0} = (\sin(\theta + \beta)(f_\epsilon^+)'(\theta), \sin(\theta + \beta)(f_\epsilon^-)'(\theta)),$$

for some β depending on the family of dilations. This is equal to the pair boundary values of the Jacobi field w_ϵ , and hence also of the Jacobi field \tilde{w}_ϵ .

In the final limit, we are dividing by the supremum of $\sin(\theta + B)(f_\epsilon^\pm)'$ and letting $\epsilon \rightarrow 0$. However, this is equivalent to taking the derivative in ϵ of this family. By the earlier definition, the derivative of f_ϵ^\pm with respect to ϵ equals the Jacobi field ψ on A_0 . This proves finally that the boundary values of the limiting Jacobi field \tilde{w} above must be

$$(10.2) \quad (\tilde{w}_0^+, \tilde{w}_0^-) = (\sin(\theta + \beta)(\psi_0^+)'(\theta), \sin(\theta + \beta)(\psi_0^-)'(\theta)).$$

Now let us choose the Jacobi field ψ generating the family A_ϵ . Expanding boundary values into their Fourier series

$$\psi_0^\pm(\theta) = \sum_{k=0}^{\infty} (a_k^\pm \cos k\theta + b_k^\pm \sin k\theta)$$

then the constraint (10.1) is equivalent to two conditions $a_1^+ + a_1^- = b_1^+ + b_1^- = 0$, and apart from these, all the other Fourier coefficients can be chosen arbitrarily (sufficiently small, of course, and so that the resulting function has the correct regularity). However, evaluating the expressions on the right in (10.2) yields a function which contains

$$\underbrace{(-a_2^\pm \cos \beta + b_2^\pm \sin \beta)}_{B_1^\pm} \cos \theta + \underbrace{(-b_2^\pm \cos \beta - a_2^\pm \sin \beta)}_{B_2^\pm} \sin \theta$$

for any choice of B_1^\pm, B_2^\pm , since these coefficients depend only on the coefficients a_2^\pm, b_2^\pm . However, this is a contradiction, since these are the leading coefficients of the element of $\tilde{w} \in \mathfrak{J}$. This proves that it is impossible that $\dim \mathfrak{J}^0(A_\epsilon) = 2$ for all ϵ .

We are reduced to the case where $\dim \mathfrak{J}^0(A_\epsilon) = 1$ for almost every ϵ . In the preceding part of the proof, it is not hard to argue slightly differently to show that in fact there cannot even exist a sequence $\epsilon_j \searrow 0$ for which $\dim \mathfrak{J}^0(A_{\epsilon_j}) = 2$ and with appropriate Fourier coefficients nonzero, so we can assume that the dimension is 1 for all $\epsilon \neq 0$. In this case we can actually still proceed almost as before. The difference is that the family of dilations D_λ generating $A_{\epsilon, \lambda}$ is no longer arbitrary. Instead, observe that the eigenspace $\mathfrak{J}^0(A_\epsilon)$ (at least along a sequence $\epsilon_j \searrow 0$) must have a limit E_0 , which is a one-dimensional subspace of $\mathfrak{J}^0(A_0)$. Since $\mathcal{D}(A_\epsilon) \rightarrow \mathfrak{J}^0(A_0)$ smoothly, we can choose a subspace $E_\epsilon \subset \mathcal{D}(A_\epsilon)$ which converges smoothly to E_0 . It is now clear that the difference vector w_ϵ defined in the earlier step of the proof may still be chosen so that its normalization has a limit as $\epsilon \rightarrow 0$. The remainder of the argument proceeds exactly as before. \square

Remark 10.2. It is not hard to see from this argument that we can produce families of solutions A_ϵ converging to A_0 which are nondegenerate and also invariant with respect to rotation by π around the axis of A_0 . These are important in one of our global existence theorems (Proposition 9.9 and Theorem 9.10).

REFERENCES

- [1] P. Collin, L. Hauswirth and H. Rosenberg. *Properly immersed minimal surface in a slab of $\mathbb{H} \times \mathbb{R}$, \mathbb{H} the hyperbolic plane*. Arch. Math. **104** (2015), 471–484.
- [2] B. Coskunuzer. *Asymptotic Plateau Problem in $\mathbb{H}^2 \times \mathbb{R}$* . Preprint arXiv:1604.01498.
- [3] B. Daniel. *Isometric immersions into $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ and applications to minimal surfaces*. Trans. A.M.S. **361** (2009), 6255–6282.

- [4] L. Hauswirth, *Minimal surfaces of Riemann type in three-dimensional product manifolds*, Pacific J. Math., 224 (2006), 91-117.
- [5] B. Kloeckner and R. Mazzeo, *On the asymptotic behavior of minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$* , Indiana Math J. **66** no. 2 (2017).
- [6] L. Mazet, M. Rodríguez and H. Rosenberg. *The Dirichlet problem for the minimal surface equation -with possible infinite boundary data- over domains in a Riemannian surface*. Proc. London Math. Soc., **102** (2011), no. 3, 985–023.
- [7] L. Mazet, M. Rodríguez and H. Rosenberg. *Periodic constant mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$* . Asian J. Math., **18** (2014), no. 5, 829–858.
- [8] R. Mazzeo, *Elliptic theory of differential edge operators, I*. Comun. Par. Diff. Eq. **16**(1991), 1616–1664.
- [9] B. Nelli and H. Rosenberg. *Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$* . Bull. Braz. Math. Soc. (N.S.) **33** (2002), no. 2, 263–292. Errata: Bull. Braz. Math. Soc. (N.S.) **38** (2007), no. 4, 661–664.
- [10] B. Nelli, R. Sa Earp and E. Toubiana. *Maximum principle and symmetry for minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$* . Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) Vol. **XIV** (2015), 387–400.
- [11] R. Sa Earp. *Parabolic and hyperbolic screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$* . J. Australian Math. Soc. **85** (2008), 113–143.
- [12] R. Sa Earp and E. Toubiana. *Screw motion surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$* . Illinois J. Math. **49** (2005), no. 4, 1323–1362.
- [13] X. Mo and R. Osserman *On the Gauss map and total curvature of complete minimal surfaces and an extension of Fujimoto’s theorem*. J. Differential Geom. **31**, no. 2 (1990), 343-355.
- [14] B. White. *The maximum principle for minimal varieties of arbitrary codimension*. Comm. Anal. Geom. **18** (2010), no. 3, 421–432.
- [15] B. White. *Controlling area blow-up in minimal or bounded mean curvature varieties*. J. Differential Geom. **102**, no. 3 (2016), 501–535.
- [16] B. White. *On the compactness theorem for embedded minimal surfaces in 3-manifolds with locally bounded area and genus*. Preprint arXiv:1503.02190v1.

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